DISSERTATIONS IN
EDUCATION,
HUMANITIES,
AND THEOLOGY

Pål Lauritzen

Conceptual and Procedural Knowledge of Mathematical Functions

Publications of the University of Eastern Finland
Dissertations in Education, Humanities, and Theology No 34

UNIVERSITY OF
EASTERN FINLAND

PÅL LAURITZEN

Conceptual and Procedural Knowledge of Mathematical Functions

Publications of the University of Eastern Finland
Dissertations in Education, Humanities, and Theology
No 34

University of Eastern Finland
Joensuu
2012

Kopijyvä Oy
Joensuu, 2012
Sarjan vastaava toimittaja: Jopi Nyman
Myynti: Itä-Suomen yliopiston kirjasto
ISBN: 978-952-61-0892-6 (print)
ISBN: 978-952-61-0893-3 (PDF)
ISSNL: 1798-5625
ISSN: 1798-5633

Lauritzen, Pål
Conceptual and Procedural Knowledge of Mathematical Functions
Joensuu: University of Eastern Finland, 2012, 158 pages
Publications of the University of Eastern Finland
Dissertations in Education, Humanities, and Theology
ISBN: 978-952-61-0892-6 (print)
ISBN: 978-952-61-0893-3 (PDF)
ISSNL: 1798-5625
ISSN: 1798-5633

ABSTRACT

Background. Function is one of the most important concepts and tools in mathematics. Its applicability depends on both conceptual and procedural knowledge. However, there are few studies of how these two knowledge types of function relate to each other and what could be an appropriate pedagogical implication. Even to find an instrument to measure the knowledge types independently from each other appears to be a hard task.
Aims. The research explored how conceptual and procedural knowledge of functions can be measured, what is the relationship between them, and how the students' ability to apply functions within economic and other mathematical tasks depends on the two types of knowledge. The outcome was related to the pedagogical philosophy applied to the study population at the upper secondary school.
Methods. Data was collected at three different stages from 476 students in economics. Confirmatory factor analysis was applied to develop tasks to measure three components: ''procedural knowledge of functions', 'conceptual knowledge of functions' and 'the ability to apply functions'. A structural equation modelling technique allowed integrating factor analysis and regression analysis into one statistical model to study relationships. Even if causal relations could not be proven, the analysis was suitable to study whether the relations suggested in the model match the sample of data.
Results. A large group of subjects showed good procedural knowledge but modest conceptual knowledge. Conceptual scores appeared even lower among those subjects who showed poor procedural knowledge. However, all students who scored high in conceptual tasks, scored also high in procedural tasks. Thus, the results support the genetic view that procedural knowledge is a necessary but not sufficient condition for conceptual knowledge. On the other hand, procedural knowledge alone seems to be insufficient for the student to be able to apply functions. The educational background of the subjects might have fostered this outcome. Interviews indicated that focus of the school teaching has been on simple procedures without links to abstract conceptual knowledge.
Conclusions. The results refer to a quite polarized pedagogy concentrating on teaching simple procedures on one hand or giving lessons on abstract definitions without appropriate links to procedural knowledge on the other hand. This might reinforce the polarization among students to so-called conceptual learners and procedurally bounded learners. To develop practical pedagogical theories, it might be important to combine the systematic analysis of conceptual and procedural knowledge of functions with a theory of knowledge structures and scaffolding within constructivist views of teaching and learning in general.

Key words: conceptual knowledge, function, genetic view, procedural knowledge.

Lauritzen, Pål
Matemaattisiin funktioihin liittyvä konseptuaalinen ja proseduraalinen tieto
Joensuu: Itä-Suomen yliopisto, 2012, 158 sivua
Publications of the University of Eastern Finland
Dissertations in Education, Humanities, and Theology
ISBN: 978-952-61-0892-6 (nid.)
ISBN: 978-952-61-0893-3 (PDF)
ISSNL: 1798-5625
ISSN: 1798-5633

TIIVISTELMÄ

Tausta. Funktio on yksi matematiikan tärkeimpiä peruskäsitteitä ja työkaluja. Sen soveltaminen edellyttää sekä konseptuaalisen tiedon että proseduraalisen tiedon hallitsemista. On vähän tutkimuksia siitä, miten näitä kahta näkökohtaa tulisi painottaa. Jo pelkästään konseptuaalisen ja proseduraalisen tiedon mittaaminen on haasteellista.
Tavoitteet. Tutkimuksessa selvitettiin, millaista konseptuaalista ja proseduraalista tietoa oppilaat liittävät funktioon käsitteeseen, miten näitä tiedon lajeja voidaan mitata, mikä on niiden keskinäinen suhde, ja miten ne vaikuttavat oppilaan kykyyn soveltaa funktioita taloustieteissä. Tulokset suhteutetaan kohdejoukon samaan lukio-opetukseen.
Menetelmät. Kohdejoukkona oli 476 taloustieteen opiskelijaa, joita mitattiin kolmessa eri vaiheessa. Konfirmatorisen faktorianalyysin avulla kehitettiin kolme eri mittaria: funktioon liittyvän proseduraalisen tiedon mittaaminen, funktioon liittyvän konseptuaalisen tiedon mittaaminen, sekä kyky soveltaa funktioita. Näiden välisten suhteiden selvittämiseksi sovellettiin strukturaalisen mallintamisen tekniikkaa.
Tulokset. Kohdejoukon enemmistöllä proseduraalinen osaaminen oli melko vahvaa, mutta konseptuaalinen osaaminen vaatimatonta. Proseduraalista tietoa huonosti hallinneilla konseptuaalisen tiedon hallinta oli vieläkin vaatimattomampaa. Sen sijaan kaikilla konseptuaalisen tiedon hallinneilla myös proseduraalisen tiedon hallinta oli korkealla tasolla. Tulokset tukevat geneettistä näkemystä, jonka mukaan proseduraalinen tieto on välttämätön, mutta ei riittävä ehto konseptuaalisen tiedon syntymiselle. Yksinomaan proseduraalinen tieto ei takaa myöskään sitä, että oppilas osaisi soveltaa funktiota, vaan hänen on hallittava myös konseptuaalinen tieto. Oppilaiden lukiossa saamalla matematiikan opetuksella oli ilmeinen vaikutus tuloksiin, sillä haastattelut paljastivat opetuksen keskittyneen yksinkertaisin proseduureihin vailla pyrkimyksiä linkittää niitä käsitteelliseen tietoon.
Johtopäätökset. Tulokset viittaavat polarisoituneeseen matematiikan opetukseen, missä yhtäältä harjoitellaan yksinkertaisia proseduureja ja toisaalta käsitellään luentomaisesti abstrakteja asioita linkittämättä niitä käyttökelpoiseen proseduraaliseen tietoon. Tämä saa luultavasti aikaan polarisoitumisen myös oppilaiden oppimistyyleissä: ns. konseptuaaliset oppijat vs. proseduureihin sidotut oppijat. Edelliset pyrkivät ymmärtämään käsitteet vailla kiinnostusta niiden soveltamiseen, kun taas jälkimmäiset opettelevat ainoastaan yksinkertaisia proseduureja pyrkimättä ymmärtämään niiden pohjana olevia käsitteitä. Kestävien ja elinvoimaisten pedagogisten käyttöteorioiden kehittäminen edellyttää matematiikan konseptuaalis-proseduraalisten tietorakenteiden systemaattista analyysiä sekä samalla sen linkittämistä tieto- ja oppimisteorioihin.

Avainsanat: geneettinen näkemys, funktio, konseptuaalinen tieto, proseduraalinen tieto.

Acknowledgement

The motivation for this dissertation developed during many years of employment as lecturer in mathematics at BI Norwegian Business School. It is written as my doctoral dissertation at the Philosophical Faculty at the University of Eastern Finland. I would like to thank Professor Lenni Haapasalo for encouragement and inspiration as my advisor. The discussions with him were always stimulating and constructive. I also thank Professor Bernd Zimmermann and Professor Jozef Hvorecky for reading the manuscript and giving valuable suggestions for improvement. Finally, I would like to express my gratitude to both my advisor and Professor Jopi Nyman, who worked a lot to help me improve the language of the thesis and its layout.

Pål Lauritzen
Oslo, August 2012

Contents

LIST OF TABLES

LIST OF FIGURES

1 Introduction

As a lecturer in mathematics I have noticed that students struggle with their courses in economics because of their lack in mathematical understanding. They do not, for example, understand the meaning of the derivative or the relation between time and interest rates even though they have learned at school to make mechanical derivations and to use a calculator. When thinking of the basic reasons behind those problems, it is especially the poor knowledge of the concept of function that might cause the most problems among students. Because the function is a typical example of a mathematical object, having both conceptual and procedural features, studying the relation of those two knowledge types in more detail was a natural choice. A recent study by Haapasalo and Kadijevich (2000) that includes a discussion of pedagogical approaches offers a solid framework theory to carry out empirical studies. Having access to a large group of students gave an opportunity to collect a large sample for the analysis to be applied in quantitative research.

In addition to analyzing students' knowledge of functions, it was interesting to find out how the pedagogical approach the students met at school could explain the findings from the statistical analysis. Would the analysis reinforce my experiences as a mathematics lecturer that most of the students focus their main attention on mechanical algorithms and procedures without aiming to understand the mathematical concepts? Designing a statistical analysis based on previous studies in mathematics education and judging the outcome through the lenses of qualitative interviews seemed to be a suitable approach to investigate this phenomenon.

After establishing the Background and Aims of the study, the quantitative and qualitative Methods are represented. The Results have been established with the statistical data and are therefore represented in detail, whilst the Conclusion part discusses the pedagogical implications of the findings in more general level, giving also suggestions for further studies.

2 Background

2.1 THE DUALITY OF MATHEMATICAL KNOWLEDGE

There is a duality of mathematical knowledge: *knowing how* vs. *knowing why*. Different labels that are applied for this polarisation are not restricted to mathematics, but seem important in questions of acquisition in general (Scheffler, 1965), (Hiebert & Lefevre, 1986). After discussing this duality by using the example of the function, the appropriate terminology to be used in this dissertation will be established.

Having appropriate knowledge of the concept of the function is probably one of the most important requirements in the study of mathematics and also within fields of research where concepts are explained in mathematical terms as in engineering, economics and finance. Functions are used to describe relationships between variables and for problem solving purposes. Dubinsky and Harel (1992) claim that the concept of the function is the single most important concept from kindergarten to graduate school. Despite agreement on the importance of functions in compulsory school, students at the bachelor’s level seems to struggle with problems involving functions. Numerous concepts within the field of economics are explained or expressed by functions represented graphically or by algebraic[1] expressions. As an example, students have problems with concepts like present value and internal interest rate in finance. Systems of equations and sequences are frequently used in finance, differentiation in social economics and so on. Examples can also be found within the field of statistics where tests based on distributions can hardly be understood without a reasonable idea that the distribution is in fact a density-function. It seems obvious that the student must be able to put the adequate meaning into the concept of a function to be able to understand economics and statistics.

The following task in derivation, as a part of a larger test, was given to 200 economics students:

Figure 2-1 represents two graphs in the same coordinate system. One belongs to the function f(x), and the other to f ′(x). Decide which of A and B that belongs to f(x) and which belongs to f′(x). Explain how you arrived at your result. You can refer to A and B in your answer. It is not necessary to sketch the graphs.

---

[1] Functions expressed in terms of polynomials or roots are denoted algebraic functions. Other functions such as exponential, logarithmic and trigonometric functions sometimes are called non-algebraic functions (Chiang & Wainwright, 2005). In the test used in this thesis most functions are algebraic.

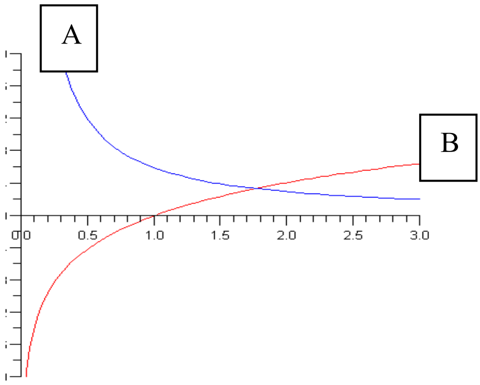


Figure 2-1. The graphs of a function and its derivative.

The intention of this task was to test to what extent the students manage to answer a question about the relationship between a function and its derivative without information of the formula for the graph. In other words, the ability to interpret the graphs functions without the underlying algebraic processes used to draw the graphs. Many students made an assumption for an algebraic expression for one of the functions. A typical answer was that A was the graph of the function $1/x$ and, since B does not look like the derivative of $1/x$, then B had to be $f(x)$. Others, who gave the correct answer, provided adequate explanations without any assumptions about processes. The example may serve as a reminder of the kind of competence students often seems to struggle with in other disciplines than mathematics, such as economics. Very often, a phenomenon is explained by a text referring to a graph, and no algebraic expression is given. The above discussion serves as a background for the rationale to look at conceptual knowledge as a factor that explains the ability to handle problems involving functions without being bound to processes.

It is a common opinion among researchers in mathematics education that the notion of numbers emerges through counting (Nantais, Herscovics, & Bergeron, 1984). Consequently, the conceptual schema is constructed through procedures. Piaget's view is that learning begins with actions on already conceptualized concepts and after the procedures are internalized, the individual will reflect on this procedure and gain new conceptual knowledge (Baker & Czarnocha, 2002). Piaget can thus be interpreted as support for the genetic view[2] (Haapasalo & Kadijevich, 2000; Sfard, 1994). The thought is that the procedures result in an outcome that needs explanation and therefore stimulates the learner to reflect on the result (Byrnes & Wasik, 1991).

The duality in conceptualisation is also referred to as procedural and conceptual knowledge (Hiebert & Lefevre, 1986), where procedural knowledge is associated with the ability to perform procedures, while conceptual knowledge relates to knowledge of relationships. Procedural knowledge relies very much on computational skills and utilisation of procedures within different representation forms. As opposed to conceptual knowledge, procedural knowledge does not require an in-depth understanding of the

[2] Genetic view: Procedural knowledge is a necessary but not sufficient condition for conceptual knowledge (see p. 18).

underlying concept. One way of distinguishing between the two is that procedural knowledge often relies on automated procedures and unconscious steps, while conceptual knowledge requires conscious thinking. Hiebert & Lefevre (1986) divide procedural knowledge in two, knowledge of forms on one side and knowledge of algorithms and rules on the other. Knowing forms means knowing the use of symbols and legal syntax. A student who possesses this aspect of procedural knowledge would be aware that the expression $f(x)=2x-1$ is acceptable, while $f3=x\ /()$ is unacceptable. Knowledge of forms does not include knowledge on how to perform calculations or interpretations of the expressions, but rather being able to separate right from wrong in use of symbols. The other aspect of procedural knowledge relates to algorithms, which are step-by-step procedures. The steps are performed in a sequential manner, and the action to be taken on each step is determined by the state of the former. Each step can be managed separately, more or less unrelated to other parts of the task. Conceptual knowledge is something that is rich in relationships and in which linking relations are as important as each piece of information itself (Hiebert & Lefevre, 1986; Hiebert & Wearne, 1986). Two categories of relations between mathematical knowledge are established. The first is the primary level, in which the conceptual knowledge consists in recognizing the relationship between two pieces of information at the same abstraction level. For example, in the case of functions, the students may understand how to draw a graph and how to calculate function values as two separate skills, but the insight that the algebraic expression and the graph represent the same mathematical concept, is the nature of conceptual knowledge at primary level.

The following small example is intended to illustrate how students typically approach mathematical problems and how their suggested solutions can make us reflect upon how they are thinking. It is not meant to give a complete picture regarding the problem of understanding mathematics, but gives an idea of how questions related to procedural or conceptual knowledge are embedded in "everyday" examples. The example addresses many issues that are discussed later in this dissertation, and hopefully gives an idea of how this study was motivated from experiences.

In an assessment, the students were given the following rational inequality:

$$\frac{2}{x-4} \le 2 \qquad (2.1)$$

Not surprisingly, many had a tendency to multiply equation (2.1) by x-4 on both sides, as shown in the left column in Figure 2-2.

Wrong solution:	Correct solution:
$2 \le 2(x-4)$ $2 \le 2x-8$ $2x \ge 10$ $x \ge 5$	$\frac{2}{x-4} - 2 \le 0$ $\frac{2}{x-4} - \frac{2(x-4)}{x-4} \le 0$ $\frac{2-2x+8}{x-4} \le 0$ $\frac{-2x+10}{x-4} \le 0$ $x < 4$ or $x \ge 5$

Figure 2-2. Different solution strategies for rational inequalities.

Our first reaction, as teachers, is that these students have some lack of knowledge, but what kind of knowledge is missing?

If we consider a student who suggests the wrong solution, what questions can be raised regarding his or her knowledge? We know that such students do not know how to solve the problem correctly. The reasons might be that they do not remember that one has to take care of different possibilities of positive or negative values when multiplying by a term, or are misled by associations to solution strategies for rational equations. Some students have problems to reflect on an answer and detect inconsistencies in the result. Maybe they are not able to detect that the properties of the answer are inconsistent with the given problem. It seems that some have not developed control mechanisms to control that the solution should meet certain properties. It might be that they are not used to asking themselves questions like: What are the properties of the sign of x-4? How does this relate to the inequality sign when both sides are multiplied with this expression? One might suspect that many students avoid reflecting on the outcome of their solution. If this is the case, maybe emphasis should be directed towards learning strategies or teaching practices.

What can we deduce from a correct solution? Does a correct solution ensure us that the student is familiar with rational expressions or is it just a confirmation that the student remembers all the procedural steps involved in the solution? How do we develop assessments to confirm that a student has a deeper understanding of the problem than just being able to perform an algorithm, and what do we mean by "deeper understanding"? A teacher would maybe explain this problem by use of a graphic solution in addition to the algebraic solution as shown in Figure 2-3.

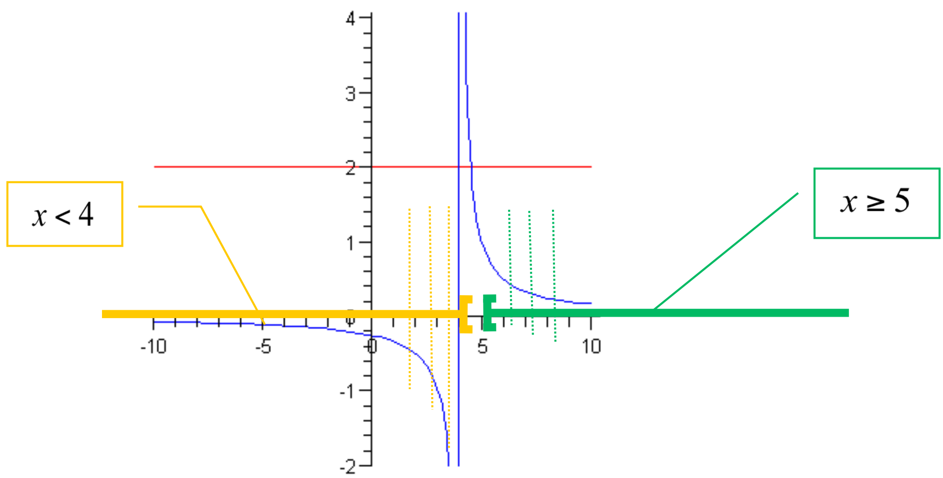


Figure 2-3. Graphic solution of inequation (2.1).

One could ask to what extent students will benefit from such an explanation if they have problems to interpret the graph or with realizing that the algebraic representation and graphic representation are just different forms of representing the same mathematical problem. The example raises questions about skills, ability to reflect on properties as well as the ability to see connections between different representation forms. Perhaps teachers assume that students understand a mathematical concept, since they perform well on algebraic skills and in reading from or drawing the graph. It might be that this type of knowledge is insufficient when it comes to relating the mathematical concept to another field as economics.

The example given in Figure 2-4 and the equation (2-2) illustrates two representations of the same mathematical problem. A student might be capable of operating on both representations without grasping the idea that this in fact represents the same phenomenon. The ability to calculate *f(x)* by taking the square of different values for *x*, and the ability to read values for the same function from the graph are consequences of having procedural knowledge, while the realization in itself that these two representations are the same, relates to conceptual knowledge.

$$f(x) = x^2 \qquad (2\text{-}2)$$

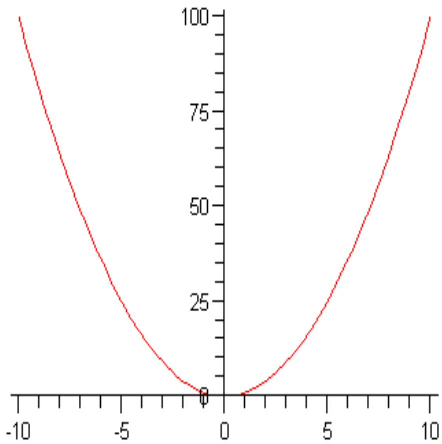


Figure 2-4. Graph of the function (2-2).

Another issue is the abstractness[3] in the two representation forms. Both are purely mathematical and are not connected to practical applications or any other kind of actual reality. In this regard, we may say that they are equal with respect to abstractness. On the other hand, they have different properties. The expression $f(x)=x^2$ carries a more exact description of the procedure to calculate function values while the graph is more suitable to immediately express properties of monotony or optimization. Similar comments can be made for other representation forms as textual and tabular representations. Especially tables, but also text, are rich in details, but less suited as means of recalling properties of a mathematical phenomenon. Also for these representation forms, the ability to understand the isomorphism between them may be regarded as a characteristic of conceptual knowledge. Hiebert and Wearne (1986) claim that mathematical incompetence often is due to absence of connection between conceptual and procedural knowledge. It is possible that a student is capable of adding two functions given by an algebraic

[3] Hiebert & Lefevre describe abstractness as the extent to which knowledge is freed from specific context.

expression and also that the same person can do the same addition if the functions are represented by graphs, but without realizing that these two operations are the same. According to Hiebert & Wearne (1986) it is the link between the different representations, earlier referred to as isomorphism, which is the core of conceptual knowledge.

Like many mathematical concepts, functions can be represented in different representation forms, such as graphs, algebraic expressions, tables or words (Janvier, 1978). Different representation forms may represent the same object. Whereas a graph is superior with respect to simplicity, when compared to a table or an algebraic expression, it does not always contain the same level of details as the two others. Hence the possibility to perform operations may depend on the representation.

One way of looking at these different representation forms is as means for communicating. Sfard (2001) presents a communicational approach for learning, where thinking can be conceptualized as a case of communication with oneself or others. She suggests that a student whose discourse is objectified has a good sense of isomorphism between different symbolic systems. When communicating with others, the student will shift back and forth between the different representations, keeping the same goal or object in mind.

Another aspect of conceptual knowledge is the reflective level where relationships are constructed at a higher level of abstraction, less tied to context. A fact is seen as a part of conceptual knowledge when the individual is able to recognize properties or connect the fact to other elements of knowledge possessed by the individual (Hiebert & Wearne, 1986). Reflections that are not connected to any particular context concern whether mathematical properties are met or not. For example, if one gets a negative function value for a non-negative function, reflective knowledge is important to detect errors caused by erroneous calculations. Of course, similar reflective abilities will play a part when we talk about applications, for example in an economic context. In this study, this characteristic will be related to the ability to apply functions.

According to Sfard (1991) structural understanding of a concept is necessary if we want to be able to use this concept to develop more complex concepts. You need to have a structural understanding of natural numbers to be able to perform operations on rational numbers. The student must be able to see the concept as one unit to be able to perform operations on the next level. This way Sfard suggests a direction of development from operational to structural in stages that she calls operation, condensation and reification. When a person has passed through these stages, he or she will have the basis to develop a structural understanding of the concept. This is a requirement that the person will be able to work at an operational level on a more advanced concept in which the first appears as a “building block”. Others (Byrnes & Wasik, 1991) support this view in claiming that procedural knowledge is dependent on existing conceptual knowledge and is achieved by use of procedures. Hence concepts can be thought of as forming a chain with respect to complexity in the same way as natural numbers are used to operate on rational numbers, which in turn are used to develop the understanding of real numbers and so on. If the success of moving one step forward in this chain of concepts depends on a structural understanding of the former concept, it should be possible to measure, to some degree, whether the student understands the concept at a structural level. An example of two concepts related in this way is that of function and differential equation. In working with differential equations functions are thought of as units that are seen as a single object or entity. In Sfard’s (1991) description of operational and structural understanding, the attention is directed towards specific mathematical concepts. Natural numbers are to be understood through phases of the previously mentioned stages of operation, condensation and reification. Reification of the concept of natural numbers must take place before one can operate on rational numbers. In this paradigm, operational understanding is a necessary condition to achieve structural understanding when the discussion is restricted to the same mathematical object. On the other hand, structural understanding of natural numbers is a necessary condition to achieve operational understanding of a more abstract concept, and in this perspective structural understanding precedes operational understanding of a more abstract[4] mathematical concept.

Herscovics & Bergeron (1983) describe a model of development of understanding with four levels; intuitive understanding, procedural understanding, abstraction and formalization which is essential in construction of conceptual schemas. As with Sfard's description of operational understanding, condensation and reification, the procedural level is a necessary condition for abstraction. When the reification has occurred, then the learner will be able to think about a concept as a unit that can be treated independently of the preceding operational steps. The concept can be treated and managed as a whole and has been interiorized into the students' knowledge base. Obviously the separation between operational and structural understanding is not as strictly distinct as the current discussion might suggest, it is rather the question of degrees of understanding.

Breidenbach et al. (1992) talk about the action and process conceptions of functions. The action conception of functions is a mental or physical manipulation of objects while the process conception of functions involves the ability to think about functions as a transformation from one kind of object, resulting in another kind of object. The description of an action conception is comparable with operational understanding and procedural knowledge as it involves such abilities as inserting numbers into algebraic expressions and calculating results. On the other hand, a person with a process conception will be able to combine the process with other processes, reverse processes and understand notions such as "one to one" and "onto". The process conception of functions addresses a deeper understanding of functions, but it does not primarily address the role of seeing relations, which is emphasized by Hiebert & Wearne (1986) as typical for conceptual knowledge. There is however a clearer parallel to structural understanding as described by Sfard (1991). Breidenbach et al. (1992) say that an action is interiorized to become a process when the action can entirely take place, or being imagined in the mind of the subject without necessarily running through all of the steps. They say, "When it becomes possible for a process to be transformed by some action, then we say that it has been encapsulated to become an object". They also emphasize the necessity of going from an object back to a process, de-encapsulation. In fact they discuss ways of thinking about functions rather than stages in a conceptual development.

Davis (1992) postulates a theory that the feeling of understanding is something you get when you manage to fit an idea into a framework of already embedded ideas. From this view, one might deduce that if the ideas that are fundamental for the idea of functions are not completely assembled, then students will have problems with the idea of a function. Assuming that the concept of function is explained to be something that describes a relation between variables, it will be difficult for the student to get a feeling what a function is without being familiar with the concept of a variable. Similarly, if the

---

[4] The term 'abstract' is used to mean 'being more difficult to relate to a context'.

student has not encapsulated the idea of a function, the student is unlikely to understand derivation.

While mathematics can be regarded as a subject that can be understood operationally or structurally, Skemp (1976) asks whether we are indeed talking about two subjects. He argues that what constitutes mathematics is not the subject matter, but a particular kind of knowledge about it. Since the two kinds of knowledge are so different, he does not only distinguish between instrumental and relational understanding, but also between instrumental mathematics and relational mathematics. Learning instrumental mathematics is about learning a number of plans on how to get from the starting point to the finishing point, while learning relational mathematics consists of building a conceptual structure from which an unlimited number of plans makes it possible to get from any starting point to any finishing point within the structure. Even if learning relational mathematics is hard, it is applicable to a variety of situations while learning instrumental mathematics is limited and makes it difficult to correct mistakes.

According to Breidenbach et al. (1992), many students do not have much understanding of the concept of function since they do not seem to be able to construct processes in their minds and use them to think about functions. Their findings suggest that the students' way of thinking about functions is influenced by use of computers, and that students tend to move from action to process conception after working with functions in a computer environment for a while. Their study revealed that the students were looking for a process, but they were not good at finding or constructing the process.

2.2 TERMINOLOGY

This dissertation relates to different subject areas, which require a clarification of the terminology. The term 'concept' is used in three different contexts. First, the mathematical concept is *function*. The term 'concept of function' will be established as the present study concerns knowledge of the mathematical concept of function. Instead of referring to 'conceptual (resp. procedural) knowledge of the concept of function', the term *conceptual (*resp. *procedural) knowledge of function* is used. Second, in factor analysis and in structural equation modelling, it is common to denote phenomena that are represented by factors as concepts. The idea is that if there is an underlying level of conceptual knowledge of functions, this will be reflected in the factor. A high score in conceptual knowledge will cause a high score on the corresponding factor that represents it. Third, and which is most important, in mathematics education the term *concept* appears as an element of 'conceptual knowledge', referring to a type of knowledge with certain characteristics. This needs a thorough discussion.

The duality *knowing how* and *knowing why* has been analysed by numerous researchers. There is not a sharp distinction between the two, but both categories have some characteristics that separate them. Dualities seem to fall into two categories with a lot of similarities (Table 2-1).

Table 2-1. Terms found in literature to name the two categories.

Operational understanding	Structural understanding	Sfard (1991)
Instrumental understanding	Relational understanding	Skemp (1976), Mellin-Olsen (1981)
Fragmented conception	Cohesive conception	Crawford et al. (1994)
Syntactic	Semantic	Nesher (1986)
Procedural knowledge	Conceptual knowledge	Hiebert & Lefevre (1986), Haapasalo & Kadijevich (2000)

The main idea of Sfard (1991) is that a mathematical concept, when understood structurally, can be seen and managed as a single unit or an object, without concern with the operations that lead to the 'structural understanding'. In this view 'operational understanding' precedes structural understanding, and reflects the student's ability to perform operations such as calculations. One could say that operational understanding reflects the ability to perform algorithms, regardless of relations to other mathematical topics and relations to previous knowledge and so on, while structural understanding has to do with relational issues. These ideas are quite similar to the distinction between instrumental understanding and relational understanding (Mellin-Olsen, 1981; R. R. Skemp, 1976). The former is seen as a learning strategy where the latter aims for rules instead of relations and structures. In the distinction of (Crawford et al., 1994) the 'fragmented knowledge' consists of knowledge about rules and formulas, whilst the 'cohesive conception' concentrates on seeing concepts as a whole.

Having assumed that 'understanding' refers to the individual's control over his/her process of knowing, Nesher (1986) made a distinction between 'learning algorithms' and 'learning towards understanding', pointing out that 'algorithmic performance' and 'understanding' can only be examined separately after the learning has been completed. Based on a long-term analysis, Haapasalo and Kadijevich (2000) suggest the following dynamical characterizations for conceptual and procedural knowledge:

- *Procedural knowledge* denotes dynamic and successful use of specific rules, algorithms or procedures within relevant representation forms. This usually requires not only knowledge of the objects being used, but also knowledge of the format and syntax required for the representational system(s) expressing them.
- *Conceptual knowledge* denotes knowledge of particular networks and a skilful "drive" along them. The elements of these networks can be concepts, rules (algorithms, procedures, etc.), and even problems (a solved problem may introduce a new concept or rule) given in various representation forms.

These characterisations depart from the conventional view of Hiebert and Lefevre (1986) that procedural knowledge would mean only rules or algorithms, represented mainly with symbolic forms, and conceptual knowledge would mean more or less formal declarative knowledge with definitions. As procedural knowledge, especially in its spontaneous informal form, can be expressed also semantically and conceptual knowledge, especially in its formal form, syntactically, the dynamic characterisation is more general and open than that of Nesher (1986). The term 'concept' is actually defined implicitly: it can be a knot, a link of a network, or even a network of other concepts. In the last case we often speak about "conceptual field".

Haapasalo and Kadijevich (2000) give a comprehensive bridging analysis of how these characterizations improve researchers' views and how they fit the modern paradigm of teaching and learning. Furthermore, they emphasize that procedural-conceptual knowledge distinction is at least person, context and content dependent. Hence a general classification between procedural and conceptual knowledge based on empirical studies seems unrealistic. Despite the attention given to the nature and relationships between procedural and conceptual knowledge, studies where procedural and conceptual knowledge is assessed from large groups of learners seem to be absent. Because conceptual and procedural knowledge cannot be measured directly, they see it appropriate to try to design conceptual tasks and procedural tasks and to study students' success in those tasks.

Because of the reasons above, it is appropriate to use the terminology of Haapasalo and Kadijevich (2000) to form the theoretical basis of the empirical studies in this dissertation. Furthermore, their theory gives a solid framework to discuss how conceptual and procedural knowledge relate to each other or depend on each other and which are the pedagogical implications. Their literature analysis reveals four views among researchers, to be represented in Chapter 2.5 with their pedagogical implications 'educational approach' vs. 'developmental approach'.

2.3 PROCESS AND PROCEPT

Gray and Tall (1994) make a distinction between a 'process', the cognitive representation of a mathematical operation, and a 'procedure', which is the algorithm for implementing a process. One process can be implemented by several procedures. For example, to calculate values for a function by putting a value of a variable into an algebraic expression and reading the functions values from a graph, can be regarded as two procedures to carry out the same process. A process does not need to be carried out. It is rather the cognitive representation of a mathematical operation that represents the process. Concepts are processes that are encapsulated, in other words, it is the process itself that is encapsulated as the concept. The concept of whole numbers is, according to Gray and Tall, strictly bound to the process of counting and it is the process of counting which is encapsulated as numbers. This is a slightly different orientation than Sfard's' (1991) theory of reification. Her claim is that children's learning of whole numbers will begin with the process of counting, the operational phase. After passing through the stages of condensation, and finally reification, the child is capable of thinking of whole numbers without being bounded to the processes from the operational stage. Even if the two views described above do not represent strictly different views on what is meant by conceptualization, it raises an interesting question. Should the process itself be regarded as a part of the structural understanding? It might be difficult to answer this question regardless of the situation we are studying.

Gray and Tall (1994) introduces the term *procept,* which represents a link between three components, symbols, process and object. The symbols are representations that serve as triggers for carrying out procedures, and also make it possible to overcome the limitations of short-term memory. In terms of functions, we might say that $f(x) = 2x$ is a symbol that represents both the object of a function as well as the process of multiplying an argument by two. The amalgam between the symbols, the process and the object, is called an elementary procept.

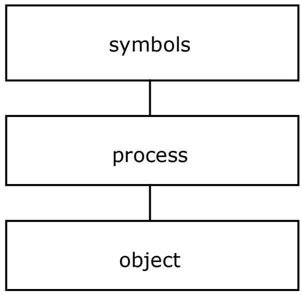


Figure 2-5. Symbols, processes and object as the components of an elementary procept.

In fact *f(x)* = *x* + *x*, is another symbolization that represents the same object. Thus a procept is in fact a class of elementary procepts as different symbols may represent the same object. Figure 2-6 visualises an idea on the how the term procept is linked to symbols, processes and object.

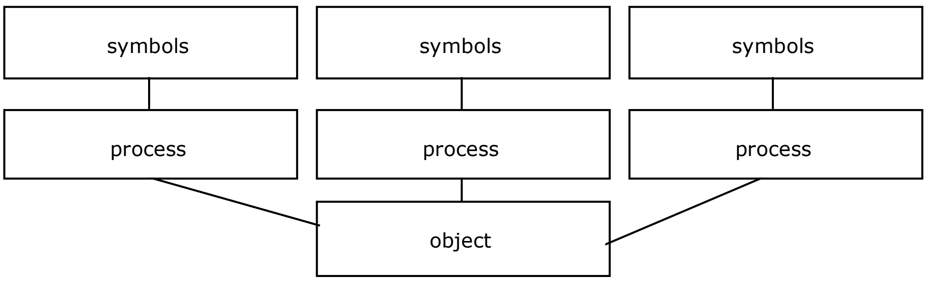


Figure 2-6. Procept consisting of several elementary procepts linked to the same object.

Even if the same symbols represent both objects and processes, some symbols are more directed towards processes as descriptions of procedures than others. If we think of a graph as a sign that represents a function, then this sign is easier to remember than an algebraic expression. In other words, it is appropriate to overcome problems with long term-memory. On the other hand, a graph is not a carrier of procedures in the same way as an algebraic expression or a table. Think of standardised normal distribution. It is easy to remember a picture of the bell shaped graph, but it is not as easy to remember the formula for the same function, or a table of values of probabilities. The formula, however, describes a much more precise algorithm for calculating probabilities.

Can the notion of procept help to clarify our view on the link between procedural and conceptual knowledge of functions? If the symbols are the glue linking processes with objects, then a representation of a function can, in a similar way, be regarded as the glue between procedural and conceptual knowledge. Instead of talking about the classes of elementary processes, one could talk about the classes of representations.

Although teachers and textbooks often focus on algebraic expressions, a text or table can represent a function, too. A text might often describe a function in terms of procedures and the functions properties. When we say that a total cost function is linear and provides values for fixed cost and variable costs, then this text can be seen as a symbol that contains information about a process, as for example how to find the total cost for a given number of units. A textbook in statistics normally contains a table of probabilities derived from a standardised normal distribution. Given the row and the column in the table, the process used to find a probability is trivially easy.

We may, in other words, regard different representation forms as a different form of symbolism to describe processes that represent one object. In this manner, the procept of function can be a useful paradigm to discuss knowledge of functions. Figure 2-7 illustrates an attempt to adapt the idea of a procept into the subject that are studied in this dissertation by regarding different representation forms as different symbolisms. For a specific function, a representation contains information of either procedures or properties, or a mix of both. One can think of algebraic representations as typical carriers of procedural information and graphic representations as typical carriers for conceptual properties.

Going back to the question on whether the process itself is a part of the object or whether the object is something that can be understood independent of the from process, we can think of children's conception of natural numbers. The process of counting to three as well as the process of measuring something with length three, both represent the object three. If the process of counting itself is encapsulated as the object, then the same can be said to be true of the process of measuring. Realizing the isomorphism between those is something more than regarding them as separate. For the idea of elementary procepts to make sense, the boxes in Figure 2-7 must be also associated in a horizontal direction, meaning that a deeper conceptual understanding is not only characterized by the encapsulation of procedures but also of seeing relations.

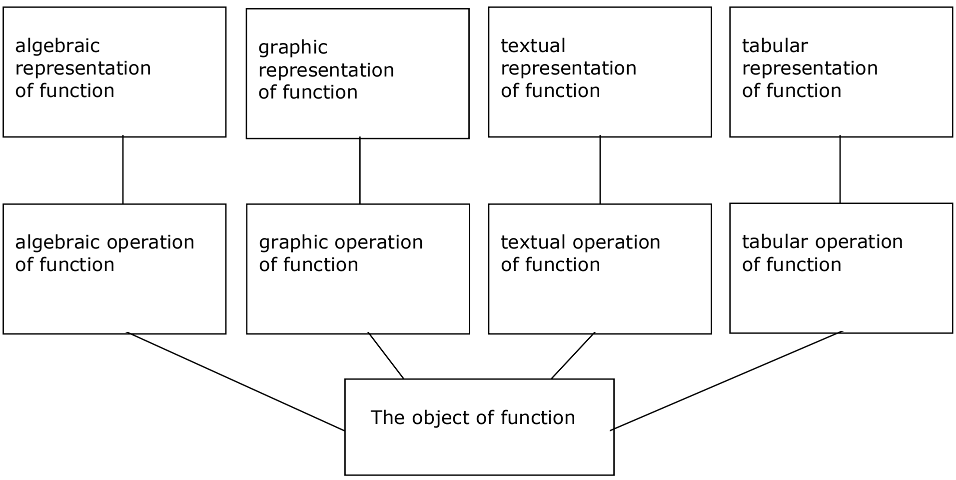


Figure 2-7. An example of how the procept of function can be seen as four elementary procepts.

2.4 THE GOAL OF LEARNING MATHEMATICS

The question "What should be the goal of mathematics teaching?" seems to trigger a debate on the importance of "conceptual knowledge" vs. "computational skills". What does it mean that computations will be made by computers and calculators? What does it mean to understand what kind of calculations to do (Resnick & Omanson, 1987)? Regardless of whether procedural knowledge is a goal in itself, how important is it to have conceptual knowledge linked to procedural knowledge? Neither procedural nor conceptual knowledge is probably the goal for a non-mathematician. His or her motivation for learning mathematics is to be able to apply mathematics in practice or maybe in another academic field. This raises the question of how different kinds of knowledge types have an effect on the ability to apply mathematics.

The next question is directed towards the consequences of teaching practices and learning strategies. It is important to reflect on whether extended knowledge on the research questions that are addressed can give some advice on how to improve teaching practices. Are skills a necessary condition for understanding? If so, how should this influence our teaching practices? Perhaps the ultimate question we should ask ourselves is how should we "teach for understanding"? A study by Kadijevich and Haapasalo (2001) shows that procedural- conceptual links can be promoted through learning activities. The considerations in Chapter 2.5 show that the two knowledge types must be somehow related in every learning process and it is the pedagogical framework theory that matters (e.g. developmental or educational approach). Not only teaching practices, but also students' approaches to learning, may have an impact on the learning outcome. There is reason to believe that some students focus on memorization of procedures rather than looking for relations. One could say that the procedural knowledge is very often what they are looking for. If this assumption is true, at least for some students, we should think in terms of how the student's attention could be drawn towards deeper conceptual knowledge.

According to Resnick and Omanson (1987) a recent claim among mathematic educators is that "there will be little need for highly practiced computational skill in the future, and that instructional focus on intentional skills is therefore misplaced". This claim addresses two questions. One question is whether the first part of the sentence is true; do we really see a tendency toward a society in which highly computational skills are redundant? The other is whether the conclusion that the focus on skills is misplaced is correct.

There is little doubt that the number of computers and calculators is increasing, but so is also the amount of calculations to be carried. A modern car today calculates fuel consumption, temperature and average speed and a lot of other things, which was not done earlier. The construction of planes, oilrigs and large buildings requires advanced calculations performed by computers. It may even be that the need for calculation skills increases. If someone is buying a mobile phone, and can choose between different offers from network vendors, how can he or she decide which one is the best without performing calculations on the cost of each alternative related to his or her need? The intention of this study is not to prove that the claim is wrong, but argue that it is at least highly questionable. Numerous researchers in mathematics education emphasize the need for both skills and conceptual knowledge.

Fischbein (1993, p. 232) describes the formal and algorithmic components of mathematics as human activities. The formal activities involve axioms, definitions, theorems and proofs, which must be components in the reasoning activities to achieve conceptual understanding. The second component, the algorithmic component, refers to skills. According to Fischbein, there is a symbiosis between those two components:

> *"There is a widespread misconception according to which, in mathematics, if you understand a system of concepts, you spontaneously become able to use them in solving the corresponding class of problems. We need skills and not only understanding, and skills can be acquired only by practical, systematic training. The reciprocal is also sometimes forgotten. Mathematical reasoning cannot be reduced to a system of solving procedures." (p. 232)*

For a student, the reasons for learning mathematics are to be able to solve everyday-problems where mathematics is involved. It is reasonable to assume that algorithmic skills alone are insufficient as a basis for applying mathematics, but also conceptual knowledge alone is probably not enough to solve mathematical problems. Fischbein suggests that the ability to apply mathematics also requires skills.

The curriculum for upper secondary education in Norway (Ministry of Education, 2000) states the demand for skills and what is referred to as knowledge in mathematics, by saying that:

> *"More and more of us find that our studies or work require the use of advanced mathematical skills or presuppose a knowledge of mathematics. Mathematical theory and practice are an integral part of modern science, technology and economics, and the subject has been essential to the development of our culture." (p.1)*

Further, the curriculum emphasizes the importance of understanding as a goal for teaching, and not drilling on the mastery of skills:

> *"Many people have the impression that mathematics consists entirely of endless drilling. Although this is an important way of teaching arithmetical skills and improving pupils' understanding of mathematical concepts, it must not become the purpose of the teaching. Mathematics is not a collection of "recipes" and algorithms for solving routine tasks, but a toolbox containing the equipment needed to solve problems that require both imagination and understanding." (p.2)*

Nesher (1986) gives an example for coping with the concept of "mean". One way of understanding it, the syntactic way, is being able to calculate means in practice. On the other hand she mentions the semantic understanding of mean as understanding the concept of the mean. The question is which approach we should use in teaching this concept. Should the students "do means" or should they be given a set of properties for the concept of mean. As examples of such rules, Nesher mentions that the mean value of a set of numbers should not be outside the range of the numbers included. The mean does not need to be among those numbers. It seems obvious that at some point, the students must "do means" to grasp the idea, but it is important that the goal of learning the mean does not only involve mastery of skills, but also to understand the properties of the mean. The control system is something that is needed to decide whether an answer is correct or not. If a person is asked to calculate the sum of two odd numbers and get an odd number as the answer, something apart from the algorithmic operation that was performed should tell him that the answer is wrong. That is, a control system with a semantic rule when internalized, can operate independent of the preceding algorithm. Even if we accept that conceptual knowledge is an important goal of teaching mathematics, the question of how these control systems and rules are learned remains.

"The shared belief among math educators so far is that one should teach for understanding since this contributes to developing the monitoring control system that the student needs in doing algorithms" (Nesher, 1986, p. 8).

The question is not whether the teaching should emphasise syntactic rules and procedures as opposed to semantics and conceptual knowledge, but how the balance should be. Even if we think that knowledge about procedures alone is insufficient for working with mathematics, they might be critical as a mean for developing a deeper conceptual knowledge. Some reflections on how procedural and conceptual knowledge influence each other during the phases of a learning process might help to clarify the way we think about this problem.

The ability to see relations and to put meaning into mathematical concepts is, from a constructivist's point of view, something that is created in the mind of the learner as opposed to the transmission view of teaching where the word and actions of the teacher carry the meaning (Cobb, 1988). It is more likely that rote learning of algorithms is better suited for transmission than the case is for development of conceptual knowledge. According to Cobb (1988, p. 89), "One of the teacher's primary responsibility is to facilitate profound cognitive restructuring and conceptual reorganizations". The constructivist view certainly points to a dilemma related to the constraints in information passed from the teacher to the learner in the sense that it is impossible for the mathematical meaning to be entirely embedded in the words or symbols that are communicated. This does not mean that teaching algorithms is superfluous, but that the teacher must be aware of the interplay between instructions on algorithms and guiding towards knowledge. The guiding could contain elements suggesting students to look for alternative solutions, pointing to limitations in the current knowledge and prevention of misconceptions.

Cobb (1988) claims that students who have constructed powerful conceptual structures will be better able to solve problems in a variety of situations a considerable long time after the learning took place. The structures are a permanent part of the student's problem solving repertoire. In this regard, the ability to apply a concept can be seen as an integrated part of conceptual knowledge. However, the ability to apply mathematical knowledge is probably the motivation or goal for learning mathematics for many students, rather than seeing knowledge as an isolated goal.

A rationale for learning mathematics is to be able to apply mathematics in an everyday situation, a professional context or to be able to learn mathematics at more advanced level. This varies among students, and neither students nor teachers can possibly predict an individual's future need for mathematics. Despite variation in their future need for mathematics, most students must develop knowledge that can be applied in unknown contexts. This view might justify that we look at the causal direction that considers conceptual knowledge as a cause for the ability to apply. On the other hand, working with functions and seeing applications of functions is probably important to build conceptual knowledge. Think of the development of control mechanisms that are needed to detect errors which is a characteristic for conceptual knowledge (Byrnes & Wasik, 1991, p. 777). Such control mechanisms will often be related to real world applications as for example when a student in economics discovers that the estimated interest rate must be too low or that a price-estimate is unrealistic. One could say that it is a goal to learn mathematics without being bound to a particular context, while on the other hand the contexts play an important role when mathematical concepts are learned.

Exams with focus on skills might hide an absence of conceptual knowledge. An example typical for students in economics is the optimization of a function with more than one variable with constraints. A frequently applied method for solving such problems is the use of Lagrange multipliers. The students are trained in the required routines, and the tasks are very similar. Also the context is typically limited to the context of cost, price, revenue and income. At the exam, students are exposed to a similar problem, and the majority of them are able to get the right solution. It is tempting to allow ourselves to believe that the students understand the mathematics conceptually, but this is probably not true. Schoenfeld (1982) says about this tendency that "To allow them, and ourselves, to believe that they 'understand' the mathematics is deceptive and fraudulent". If there is a general concern that the focus of teaching is too much directed towards skills, a reorientation towards teaching for understanding must be accomplished by a similar focus in assessments.

Assuming it is true that we need less computational skills in the future, one cannot deduce that instructional focus on intentional skills is misplaced. It might be that these skills are a necessary condition for conceptual knowledge. In other words, even if the goal of mathematics teaching is to help the pupils to achieve conceptual knowledge, the procedural knowledge may very well be a tool to reach that goal.

To summarize this part of the discussion, it seems reasonable to say that we need computational skills as well as conceptual knowledge, but that today's teaching and assessments do not pay enough attention on conceptual knowledge. Procedures are often well defined and it is relatively easy to see if they are carried out adequately, but conceptual knowledge is richer in relationships (Gray & Tall, 1994) and necessary to be able to apply mathematics and reflect on the results. This research hypothesizes that both procedural and conceptual knowledge are required to apply mathematics successfully. If we accept that both are important, the question is rather how to teach. The development of a sound teaching strategy should be based on an awareness of how procedural and conceptual knowledge are related to each other.

2.5 LINKING PROCEDURAL AND CONCEPTUAL KNOWLEDGE

The discussion on procedural and conceptual knowledge represents different views in which one type of knowledge is a necessary or maybe sufficient condition for the other, or whether they relate to each other at all. It seems to be generally accepted that it is not strictly one way or the other. It is hard to operate on functions without knowing the concept of function, but it is also unlikely that one is able to put meaning into functions without being able to operate on them. Nevertheless, it is of interest so see if there are tendencies that are more dominant than others.

Mathematical knowledge consists of both procedural and conceptual knowledge, and "Linking conceptual and procedural knowledge would have many advantages for acquiring and using procedural knowledge" (Hiebert & Lefevre, 1986). Perhaps it is possible to possess one of them, but that this is incomplete in the sense that one can have a good feel for mathematics without being able to perform calculations or one can be able to calculate answers without understanding their meaning. The statement indicates that it is the link between the two knowledge types that is important for us to be able to apply mathematics. This addresses the core of the problem in this dissertation. What are the advantages of being able to link conceptual and procedural knowledge? First, it allows an individual to reason on the meaning of the object, rather than reasoning through an intermediate language. Second, the ability to select and effectively utilize procedures will benefit from this link (Hiebert & Lefevre, 1986). Third, conceptual knowledge will increase the ability to detect a wrong use of procedures or inappropriate procedures in a given situation. Also, conceptual knowledge provides the ability to monitor the outcome of a mathematical operation, i.e. reflect on the answer. It provides a basis for building control mechanisms for detecting whether an answer makes sense.

The distinction between procedural and conceptual knowledge can be beneficial in the sense that they serve different cognitive functions. Byrnes and Wasik (1991) claim that conceptual knowledge imposes organization of experiences and relate things, while procedural knowledge is considered a mean to achieve certain goals. Assuming that both procedural and conceptual knowledge is important in itself, and therefore a goal of mathematical learning, it is natural to ask how knowledge types are related, also with respect to causality. Theory suggests a variety of possible links between procedural and conceptual knowledge, but it is hard to prove these dependencies. The word "prove" seems highly unrealistic if one thinks in terms of drawing conclusions with absolute certainty. Instead one can think of proofs in statistical terms, where something is proved if the data supports the theory well enough. Literature analysis of (Haapasalo & Kadijevich, 2000) reveals four views on causal dependencies between conceptual knowledge (abbreviated by ***C***) and procedural knowledge (abbreviated by ***P***) related to the learning process (Table 2-2).

*Table 2-2. Views on the relation between procedural (**P**) and conceptual (**C**) knowledge.*

Genetic view: ***P*** is a necessary but not sufficient condition for ***C*** (Kline (1980), Kitcher (1983),Vergnaud (1990), Gray & Tall (1993) and Sfard (1994)).
Dynamic interaction view: ***C*** is a necessary but not sufficient condition for ***P*** (Byrnes & Wasik (1991)).
Simultaneous activation view: ***P*** is a necessary and sufficient condition for ***C*** (Hiebert (1986), Byrnes & Wasik (1991) and Haapasalo (1993)).
Inactivation view: ***P*** and ***C*** are not related (Nesher (1986) and Resnick & Omanson (1987)).

The *genetic view* states that procedural knowledge is a necessary but not sufficient condition for conceptual knowledge. This seems to be a view that is supported by many researchers in the sense that they describe concepts as processes that are being encapsulated (Dubinsky, 1991; Gray & Tall, 1994; Kaput, 1982; Sfard, 1991). Procedures are, in this view, seen as a fundamental part of conceptual development and that a cognitive shift takes place when the concept is encapsulated as an object. The procedural phase occurs prior to the conceptual phase indicating a causal direction. In other words procedural knowledge is seen as a necessary condition for conceptual knowledge, but possibly not a sufficient one. Sfard (1991) argues that operational concept formation occurs prior to structural concept formation, whether we regard this in a historical view or we regard the individual. The historical view assumes that an individual's development of a mathematical concept follows in the same order as the development of the concept in history. As an example she mentions the notion of numbers, which originates with the process of counting. An objectified discourse of for example whole numbers is required that one is able to operate on rational numbers, which is the same sequence the development has taken place historically. Even if an individual develops skills, he or she does not necessarily have to develop conceptual knowledge according to this view.

A contrast to the genetic view is the *dynamic interaction* view. The argument that supports the dynamic interaction view, i.e. that conceptual knowledge is a necessary, but not sufficient condition for procedural knowledge, is that conceptual knowledge makes the construction of procedures possible (Byrnes & Wasik, 1991). One reason is the development of control mechanisms, which are helpful to detect procedural errors. Byrnes and Wasik claim that computational errors are caused by the fact that mathematical symbols are meaningless to many pupils and that procedures are meaningful only if they can be related to objects. They also argue that conceptual knowledge will contribute to the detection of computational errors. If a pupil adds two fractions erroneously by adding numerator with numerator and denominator with denominator, a well-developed (conceptual) knowledge of magnitude of rational numbers might serve as a mean for detecting computational errors and cause the student to redo the calculation. One could ask whether improved skill in adding fractions, as in this case, was caused directly by conceptual knowledge or the procedure repetition.

To verify the *simultaneous activation* view might seem to be a rather ambitious task, since this view not only claims that procedural knowledge is a necessary, but also a sufficient condition for conceptual knowledge. In fact it means that other explanatory variables for conceptual knowledge should be considered redundant. However, the rationale for the formulation of this view is probably not grounded in proofs of redundancy. Rather, it originates from tests showing that computational errors are caused by an impoverished conceptual knowledgebase (Byrnes & Wasik, 1991). In other words, the studies imply that lack of conceptual knowledge causes lack of procedural knowledge. As the term simultaneous suggests, the development of procedural and conceptual knowledge is in some sense considered to be parallel in time. When working with procedures, conceptual knowledge will be used and further developed. Thus, besides trying to find empirical verification for the causal relation regarding student scores in conceptual vs. procedural tasks, it is appropriate to study the pedagogical power of the simultaneous activation view. Having realised that procedural links may be established through learning activities requiring, among others, production rules utilisation and multiple representation transformation, Kadijevich and Haapasalo (2001) represent two constructivist technology-based environments concentrating on finding more or less systematic instructional models. The other one of those models is utilising CAL software, developed within the so-called MODEM project (see Haapasalo 1993). Eronen and Haapasalo (2010) found out that utilising simultaneous activation by using modern graphic calculator could shift the learning of mathematics from using textbooks to playing with hands on –technology. Dynamical geometry software, especially combined with algebra and robotics, can do the same, as Haapasalo and Samuels (2011) describe in their comprehensive article.

The *inactivation view*, suggests that procedural and conceptual knowledge are not related. Even if studies normally are not designed to prove the absence of such relations, some studies suggest weaker relationships than one might expect. One argument in favour of this view is that some students may have a high level of conceptual knowledge, but lack in procedural skills. With others it might be the opposite; they may have a high level of procedural skills and a low level of conceptual knowledge. Resnick and Omanson

(1987) conducted a study aimed at testing whether procedural learning would be more successful if it was well-grounded in mathematical principles. In other words, they tested if better conceptual knowledge would cause pupils to perform better on skill-oriented tasks. Surprisingly, the study revealed that children seemed to fail to apply their knowledge of principles when performing subtraction calculations. Zucker (1984) could not find a significant correlation between algorithmic performance and understanding in decimals, despite a sample size of 270 pupils. One cannot conclude that such a relation does not exist, but it is surprising that the study did not result in a significant correlation between the two types of knowledge. Even if these studies support the inactivation view, the majority of studies regarding relationships between procedural and conceptual knowledge support the existence of such a relationship.

The first three views all describe some causal dependencies between procedural and conceptual knowledge. Even if the different views seem to contradict each other at first sight, they should not be regarded as competing models that claim to be generally true. One should rather struggle to investigate the learning process in pieces in which each piece is studied with a particular view. Let us assume that procedural knowledge and conceptual knowledge are mutually dependent on each other. One can consider the learning process as taking place in time where the learner shifts between using procedures to develop conceptual knowledge and vice versa. The first piece of this process can be regarded with the simultaneous activation view or genetic view, and the other with the dynamic interaction view. Another way of dividing a learning process into pieces can be to look at how an objectified discourse of one object is required so that we are able to operate on more advanced objects in which the first type of object is a "building block".

Next follows a consideration of the pedagogical implications of the three first views in more detail. Haapasalo and Kadijevich (2000) define two pedagogical approaches, the *developmental approach* and the *educational approach*. The first one is supported by the genetic view and the simultaneous activation view, whilst the latter is supported by the dynamic interaction view and the simultaneous activation view. Thus, the approaches imply different instructional interpretations but simultaneous activation can be utilised within the both approaches.

The developmental approach is based on the idea that procedural knowledge precedes conceptual knowledge. This is supported by Sfard's (1991) view that development takes place through stages: operation, condensation and reification. The idea is that the procedural stages must have been passed through for reification to take place. Reification is in a way an objectification of the mathematical concept. According to Vygotsky, interpersonal social activities will be internalized as interpersonal actions that must be imposed on the learner by built in methods of thinking (Haapasalo & Kadijevich, 2000). In a mathematical learning process, these methods can be algorithms or procedures.

The educational approach assumes that procedural knowledge is enabled by conceptual knowledge. Carpenter (1986) claims that children should have an idea of the fraction or a reasonable concept of addition and that these ideas are foundations for performing procedures. Without the ideas, the procedures are carried out with symbols that make no sense. Many textbooks in mathematics start the introduction of a topic by giving a definition and follow it up by examples. In these cases, the concept-definition comes first, before the operations are described. Even if the learning material guides the student to change between procedural tasks and focus conceptual qualities, the underlying assumption often seems to be that the very first presentation of a concept should be directed towards its definition or its properties. There are longitudinal studies that showed a development where conceptual knowledge precedes procedural knowledge in time (Byrnes & Wasik, 1991). In his long-term MODEM –project, Haapasalo found that a more or less systematic model can be utilised to promote conceptual knowledge by emphasizing multiple representations (Haapasalo, 1993; Kadijevich & Haapasalo, 2001).

The teacher's role may also have an important impact. Haapasalo (1993) gives evidence that traditional mathematics teaching is polarized to apply the above-mentioned approaches in exaggerated way. Teachers, especially on preliminary grades, go for procedures and trust that conceptual knowledge would appear as a consequence of that. On the other hand, especially on upper grades, mathematical concepts are defined at first, followed by drilling and practicing. Both approaches seem to lead not only to poor cognitive results but also to negative mathematical beliefs among students. As Eronen and Haapasalo (2010) show, the developmental approach and the educational approach can be combined within the quasi-systematic model of Haapasalo, for example by utilising the theory of Vygotsky and modern technology.

Ma (1999) suggests that the teacher's knowledge level decides their teaching strategy. Many procedurally oriented teachers intended to teach for conceptual knowledge, but this is not reflected in their teaching. When pupils are failing to solve problems, a lot of teachers explain details of the algorithms rather than addressing understanding. In other words, some teachers seem to approach the students' problem as if it were caused by the fact that the students have forgotten an algorithm rather than explaining the mathematical meaning of the procedures.

Finally, it is possible that the different views vary with topics. A student might for example approach matrix algebra with a focus on procedures to begin with, while the same student may start to learn about the derivative with an emphasis on properties and definitions. Figure 2-8 illustrates an example of how different pieces of mathematical education can be regarded with a combination of the views in mind.

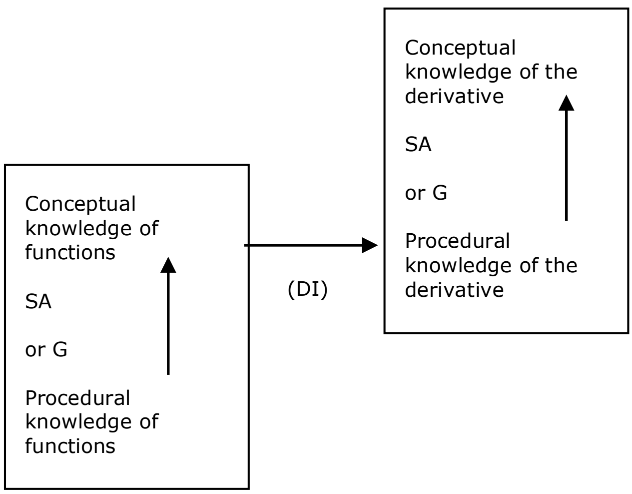


Figure 2-8. Simultaneous activation view (SA), Genetic view (G) and Dynamic Interaction view (DI) in different learning stages, indicated by arrows.

The idea is from Sfard (1991) where a similar link is described in the case of concepts in general. The concepts of function and the derivative of functions are used in the example. If a student's development proceeds as steps, each indicated by an arrow, then each step can be regarded in different views. One could assume that procedural knowledge of functions is a condition for conceptual knowledge of functions, in accordance with the simultaneous activation view or the genetic view. This does not mean that other causes for conceptual knowledge are neglected, but it is a question of balancing the consideration of keeping the model simple and not leaving out the important variables.

On the other hand, when we look at the horizontal arrow in Figure 2-8, we consider conceptual knowledge of functions to be a necessary condition for procedural knowledge of the derivative of functions. In that case conceptual knowledge of a concept at one level is a condition for procedural knowledge at a more advanced level. It may be incorrect to categorize this into the dynamic interaction view (hence the parenthesis), but nevertheless it is important to distinguish between knowledge types within a concept (vertical arrow) from knowledge of different concepts (horizontal arrow).

2.6 KNOWLEDGE AND APPLYING

The dichotomy in "knowledge types", regardless of how it has been labelled, has been used to address different aspects with respect to their nature and their relationship. These aspects provide premises for the discussion about what kind of knowledge is important and how this influences in the learning process. The considerations above hopefully show that instead of speaking about "levels of knowledge" it might be more appropriate to speak about "how a student scores in procedural or conceptual task types", if those tasks can be defined and designed under given pedagogical conditions.

Applying mathematics in general or functions in particular, could be regarded as an integrated part of conceptual knowledge. In other words, one could say that being able to apply mathematics in problem solving is one of the characteristics for conceptual knowledge. Another approach is to regard the ability to apply as a separate phenomenon. Cobb (1988) claims that students' abilities to solve problems in a wide variety of situations depend on their conceptual structures. He refers to situations that may include mathematical tasks that are superior to the one they have conceptualized. If we interpret this in relation to functions, we can say that students' abilities to solve problems about derivation of functions depend on their conceptualization of the concept of function. At least if "solving problems about derivation", is understood to include skill oriented tasks, then we see a parallel to Sfard's (1994) theory when she says that the student should have an objectified discourse of one concept to move on to the operational phase of a more advanced one. One example is the conceptualization of whole numbers that must be understood conceptually before it can be applied to operate on fractions. Like whole numbers are seen as a building block for fractions, functions can be seen as a building-block for differentiation.

Duffy and Jonassen (1992) introduce a notion of understanding that they call the 'performance perspective on understanding'. They distinguish between 'understanding' as 'deeper understanding' and knowledge as the mastery of skills. The performance perspective says that understanding a topic is a matter of being able to perform in a variety of thought demanding ways with the topic as for example representing a topic in a new way or being able to apply a concept. Knowledge means that you can take in and maybe replicate what you have read or heard, but understanding requires that you are able to interpret and put new meaning into what you have read or heard. It is also worth commenting that the ability to apply is included in the concept of understanding as opposed to seeing it as a separate concept. From this perspective, one could argue that the ability to apply functions could be seen as an integrated part of conceptual knowledge of functions. It is beneficial to separate the two into two measures. Applications of functions can be found in a variety of disciplines outside mathematics where economics is one of them. If the ability to apply a topic like the concept of functions is to be included in conceptual knowledge of functions, it would have to cover so many aspects that it would be hard to measure.

It seems reasonable to separate conceptual knowledge of functions from the ability to apply functions in solving problems that are not purely mathematical, such as problems within economics. Also in cases of applications, some of the problems are pure repetitions of routines, while others are of a more conceptual nature, requiring relational considerations. The following questions were given to students in the first year of an economics class.

> You borrow 800 000 NOK at 4,7 % interest rate per year. The loan is an annuity-loan and will be paid off in 15 yearly payments.
>
> What is the yearly payment when the first payment starts after one year?
>
> What is the outstanding balance immediately after the fifth payment?

Figure 2-9. Task from the pilot study about an economic problem.

Even if the task is not purely mathematical in the sense that it addresses an economic problem, question *a* cannot be said to measure much more than procedural knowledge. The question is solved using a well-known formula for annuity. When the first payment starts after one year, the formula can be applied directly, without any adjustments.

On the other hand, question *b* requires the student to choose an appropriate solution strategy from several alternatives. The easiest strategy could be to calculate the present value of the ten remaining payments. In fact this solution would be so similar to the one in question *a* that one may claim that question *b* also measures procedural knowledge. However, only a minority of the students managed to get the correct answer. In other words, the problem was not to calculate the answer given that the solution strategy was chosen, but it was to find an appropriate strategy. It is likely that some students are able to use prescribed methods to solve the tasks that are similar to those in which the method was taught, but the question of similarity is not a question of whether a task is similar or not similar to another one. It is, rather, a question of the degree of similarity. The solution of a problem may partly depend on already known procedures, while others may require a more in-depth understanding.

From an analytical point of view, it is advantageous to distinguish between procedural and conceptual knowledge of functions, and the ability to apply functions to study to what extent the ability to apply stems from the procedural knowledge of functions and to what extent it stems from the conceptual knowledge of functions. The structure in Figure 2-10 is suitable to study how procedural and conceptual knowledge of functions influences the ability to apply the functions separately.

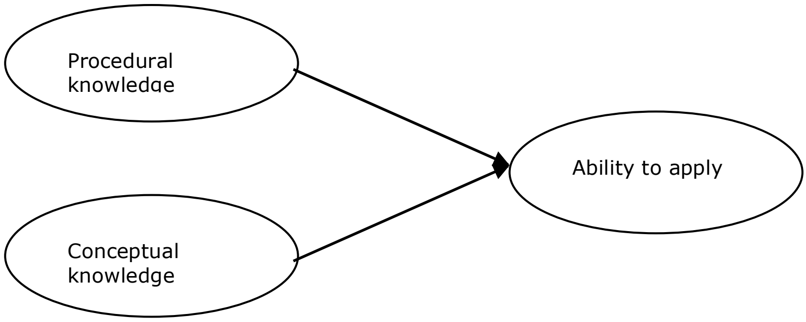


Figure 2-10. The ability to apply functions depending on both procedural and conceptual knowledge.

2.7 LEARNING AND TEACHING APPROACHES

Most of the research in mathematics education is focused on younger children, while participants of the test for this project are students mainly in their early twenties. Habits and routines for working with mathematics are shaped during one's years in school. Since teachers are different in their orientation to teaching with respect to skills and understanding, one might expect that pupils have established different ways of meeting a learning situation.

A study by Järvelä and Haapasalo (2005) suggests that there are three types of learners, the *conceptually oriented learner,* the *procedurally oriented learner* and the *procedurally bounded learner.* While the conceptually oriented learner advances from conceptual knowledge to procedural knowledge, the procedural oriented learner advances from procedural knowledge to conceptual knowledge if instructions and learning environment are tailored to establish these procedural links. The procedurally bounded learners are solely focused on procedures, and often without development towards conceptual learning. One factor that might explain this is the kind of teaching and instruction the students have met. This obviously depends on the teachers of each individual student.

Students seem to apply different approaches to learning in a given learning context (Entwistle & Tait, 1990), such as a course in mathematics. Some focus their attention toward memorizing facts with less attention to understanding principles and concepts. Others are trying to relate new knowledge to their previous knowledgebase, trying to see relations and understand theories. Marton and Säljö (1976) and Entwistle and Tait (1990) have in different but related studies identified two contrasting approaches toward learning and studying. The first one is described as a deep approach to learning, where the course participant looks for meaning and processes the text in a holistic way. The deep approach refers to a focus on the significant, to relate previous knowledge to new knowledge, to relate knowledge from different courses, to relate theoretical ideas to everyday experience, to relate and distinguish evidence and argument, and to organize and structure content into a coherent whole. The second approach is presented as a surface approach to learning, where the students pay attention to keywords in an atomistic way and focus on unrelated parts of the task. Typical for the surface approach is memorization of information for assessments. The students associate facts and concepts unreflectively, often failing to distinguish principles from examples. They are treating the task as an external imposition and try to avoid failure by means of rote learning. This distinction between learning approaches is in many regards similar to the distinction between procedural and conceptual knowledge. Biggs (1993) also argues for the existence of a third approach; this is a strategic approach dominated by the motive to achieve the best grades possible, by organization of time and learning environment. However, the distinction between the strategic approach and the surface approach is unclear, and the two approaches are often treated as one approach. For simplicity, it is therefore sufficient to only regard deep and surface learning approaches. Some denote approaches to learning as learning styles, as if they are personal capabilities or personal styles independent of situation or context (Marton & Säljö, 1976), while others claim that they are related to the specific learning context (Entwistle & Tait, 1990; Newble & Clarke, 1987). According to Biggs (1993, p. 17) the truth lies somewhere in the middle, that preferences depend both on students and on learning context. Even if an individual, in a given learning situation, has a combination of both approaches, students often seem to prefer one in favour of the other.

Students' approaches to learning may have been affected by their beliefs in what mathematical learning is. In the National Assessment of Educational Progress (1983) nine out of ten students agreed with the statement that "there is always a rule to follow in solving mathematics problems". Not only do students seem to believe that such a rule exists, but they also seem to think that there is only one way to solve a given problem. The reason for this attitude is likely to be their experiences through years in school, being exposed to situations where teachers have demonstrated procedures and given the students numerous examples of carrying out the same procedures, until they master the technique. If students and teachers regard a mathematical topic as being understood by the student when the mastery of skills is established, and then continue to work with the next topic, understanding of the next topic will obviously suffer. As an example, a student can learn techniques on how to calculate values of a function, given different values for the argument. This may lead many students to believe that mathematics is something that should be memorized rather than search for meaning. Suppose this procedure is repeated over and over again for different functions that are given algebraically, and that little effort is spent on reflections on solutions or on relational issues to graphic representations or other relational issues, then some students might believe that the topic has been learned and that the student has the necessary background to learn about the derivative.

It is not surprising that some students do not look for such things as relations between different solution strategies or reflect on them with respect to appropriateness. Since students' approaches to learning are determined by their expectations and beliefs, one can assume that their school experiences have formed their beliefs.

The other issue is the strategy the teachers choose to use when they teach mathematics. As with learning approaches, teachers will also tend to have different styles or strategies, and it is reasonable to expect personal and contextual variations. To improve teaching, we have to study how students learn and apply this knowledge in our teaching. It is not a trivial task to change ones teaching strategy even if new knowledge on students' conception is attained. Ma (1999) compared American and Chinese teachers and observed the teaching of teachers who had conceptual knowledge and those who had not. An important finding in her study was that the knowledge level of the teacher determined the teachers learning strategy. Not surprisingly, the teachers with lower subject matter knowledge were skill oriented in their teaching strategy. The study revealed that the teaching in China was more oriented toward conceptual knowledge. In China there is an attitude towards teaching problems in multiple ways and also to organize pieces of mathematical knowledge into "knowledge packages". They emphasize the importance of relating new knowledge to previous knowledge. Ma observed that these characteristics were reflected in the teacher's approaches. Obviously, it is a necessary condition for teaching for conceptual knowledge that the teacher herself has sufficient conceptual knowledge, but this does not mean that it is a sufficient condition. It is important for the teacher to be aware of the "nature" of conceptual knowledge to teach for deeper understanding. The study also revealed that American teachers were more procedurally oriented in their way of teaching. While the majority of the Chinese teachers were conceptually oriented in their teaching strategy, only a minority of the US teachers fell into this category, dependent on the topic they were teaching. Respectively 14% of the Chinese and 77% of the US teachers displayed only procedural knowledge when teaching subtraction with regrouping. Almost all the teachers with a deeper knowledge of the subject were conceptually oriented in their teaching strategy, despite their possible choice of a skill-oriented strategy. The procedurally oriented teachers tended to consider pupils' mistakes as a problem of not knowing or remembering a procedure, while the conceptually oriented teacher had a different teaching strategy, explaining the rationale of the problem and separating problems into subproblems. They also spent much more time on reflection, letting the pupils discuss and explain why they did what they did. While Sfard's theory suggests that structural understanding develops from operational understanding, the conceptually oriented teachers in Ma's study are aimed at building up conceptual knowledge in all phases of their teaching. Of course, in this approach, procedures are important. It is hard to imagine how a discussion of why a particular procedure was followed could take place without knowing how to perform the procedure.

According to Ma (1999) there is a tendency that, despite teachers' desire to teach for conceptual knowledge, many of them focus on the mastery of skills. One can think of several reasons for this. Some teachers may have lack of conceptual knowledge, while others may believe that skills are the goal of learning mathematics. A third factor can be a lack of consciousness on how to teach for conceptual knowledge. A further possibility is that some teachers may believe that if teaching focuses on mastery of skills, conceptual knowledge will develop by itself. If teaching is biased towards mastery of skills, the challenge is how to teach for conceptual knowledge. Even if there is an agreement that there is a link between procedural and conceptual knowledge, it is not well established how instruction should be designed to create this link (Hiebert & Wearne, 1986, p. 199). Perhaps the educational system is too much based on the tradition that students are evaluated on whether they are able to reproduce what the teacher tells them. Philip Jackson (1996) says that the public view on schools follows a "mimetic tradition", seeing knowledge as something which is transferred from the teacher to the learner.

The idea of learning for understanding is not a new idea. Berger and Luckman (1996) argued that each human being must construct meaning. The idea of constructivism means that the student must be able to reflect and make sense of what the teacher says and a cognitive development must take place in each student. There seems to be a consensus among many researchers in mathematics education that knowledge is something that is constructed by the learner. If we accept that mathematical ideas cannot be transmitted by words or carried from the teacher to the student by mathematical symbols, then the role of the teacher will be to guide the student in a direction that leads the student to construct knowledge by reorganizing of his or her cognitive structures. According to Cobb (1988), the challenge for the constructivist teacher is how to account for successful communication being aware that the meaning cannot be wholly embedded in the words or actions of the teacher. The constructivist teacher would perhaps try to guide the student to construct knowledge by posing appropriate questions, leading the student away from misconceptions and giving guidance in accordance with the student's cognitive development.

Schoenfeld (1982, p. 345) presents what he calls a pedagogical imperative:

> *"If one hopes for students to achieve the goals specified here – in particular, to develop the appropriate mathematical habits and dispositions of interpretation and sense-making as well as the appropriately mathematical modes of thought – then the communities of practice in which they learn mathematics must reflect and support those ways of thinking. That is, classrooms must be communities in which mathematical sense-making, of the kind we hope to have students develop, is practiced."*

The goals he is talking about are outlined in the *Source Book for College Mathematic Teaching* (Schoenfeld, 1990) where it is underlined that instructions should aim at developing conceptual understanding rather than mechanical skills. Students should be able to work in an environment where they can work with problems in an explorative way, to find patterns and develop a feeling on how things work with emphasis on structural relationships. The imperative reminds us that a reorientation towards conceptual understanding should be accomplished by requirements for teaching practices and learning environments.

Davis (1992) differentiates between what he calls "Previous view" and "Newly emerging view" on mathematics education and suggests that we witness a new way of thinking about doing mathematics. The "Previous view" regards the point of learning mathematics as learning facts and algorithms. In this view, mathematical knowledge is constructed from words and syntactic rules. Consequently, memorization plays an important role and assessments are used to find out how students can reproduce what they have memorized. On the other hand, the "Newly emerging view" emphasizes that the real essence of learning mathematics takes place in the students' mind. In this view, the mental representations are constructed by the student, and words can only be used to guide the construction of these representations. Maybe the traditional focus on procedural skills in assessments has led teachers to focus on "how" to do mathematics, and in this way stimulated teachers may follow a transmission teaching strategy. In other words, there might be an interrelation between teaching strategies and the type of knowledge that is emphasized. Cobb (1988, pp. 88-89) claims that:

> *"… the classroom situation is ripe for miscommunication when the teacher possesses structures and can "see" mathematical objects that the learners are yet to construct."*

Both Davis and Cobb are talking about changes in our view on how mathematics should be taught, but much of the rationale for this reorientation seems to be connected to a reorientation on what kind of mathematical understanding the students need. We might say that teaching skills can be accomplished, at least to a certain extent, by telling and showing how to do mathematics, but this is a method that might be insufficient for teaching deeper conceptual knowledge.

In most cases, teaching strategies will probably combine elements from transmission and construction, but it might be that traditionally too much emphasis has been placed on transmission. The distinction between the constructivist view and the view that learning takes place through transmission raises some questions on teaching strategies. If attention is towards rote learning of algorithms and how to carry out procedures, the teacher can demonstrate a lot of techniques. In these situations, less attention might be directed towards discussions on the appropriateness of the procedures or reflections on relational issues. If a teacher tries to teach a student how to do a polynomial division, the syntactic rules can be described and several examples can be demonstrated to the student. As such, it might be that the teaching of skills is often accomplished by a transmission teaching strategy. On the other hand, the development of a deeper conceptual knowledge may suffer when such a teaching approach is applied.

The question of how we should teach is relevant for many disciplines, but it is the intention of the present dissertation to study this in relation to mathematics. To be even more specific, the question is how we can teach for better understanding of functions. A project at the Harvard Graduate School of Education (Kickbusch, 1996) provided a framework of four concepts that might be helpful for teachers when they prepare to teach for understanding. The four concepts are called "generative topics", "understanding goals", understanding performances" and "on-going assessment". The framework is suitable for teaching topics that are called generative topics. One could say that the idea here is to limit the focus to a specific topic when planning for teaching. A generative topic is a topic which is central to the discipline and which is connected to other topics within and outside the discipline. There is no doubt that functions are central to mathematics. In this study, the ability to apply functions has included derivation, a topic within the discipline of mathematics, while economic applications represent connections to economics. Understanding goals and understanding performances refers to the identification of goals for each topic and how each goal should be accomplished by performances that the student needs to work with. These goals should be stated and they serve as the focus of instruction and should be limited to the actual topic in a way that they are understandable for both teachers and students.

2.8 RESEARCH CONSIDERATIONS

What kind of "result" might come out of a study in mathematics education? Niss (1998) has given a definition on research in mathematics didactics that might help to clarify what a result means in this context:

> *"The didactics of mathematics, alias the science of mathematics education, is the scientific scholarly field of research and development, which aims at identifying, characterizing and understanding phenomena and processes actually or potentially involved in the teaching and learning of mathematics at any educational level." (pp. 4-5)*

If we focus on identifying, characterising and understanding phenomena, what kind of phenomena or problems are we talking about? How can they be described or characterised? What kind of methods can we use to achieve a better understanding of these phenomena? Should the focus be on the ability to apply mathematics rather than to understand mathematics? These are two sides of the same coin. Noss (1998) claims that:

> *"It is hard to understand a mathematical idea until you have used it, until you have seen its connection with other mathematical ideas, and possible applications." (p. 10)*

but also:

> *"On the other hand, it is hard to use a mathematical idea until you have understood it, and consequently the ability to apply a mathematical concept is essential." (p. 10)*

The quote from Noss addresses the main purpose of this project, which is to study how a mathematical concept is understood at different levels, the relation between different knowledge types and the ability to apply the concept. Studies in mathematics education that have been occupied with similar questions are used as a rationale to set up a hypothesis for a statistical model. Structural equation modelling technique is applied to be able to study all aspects of the research questions within the frame of one model.

3 Aims

The research is aiming at exploring economics students' conceptualization of functions, and to investigate relationship between procedural and conceptual knowledge of functions. Another goal of the study is to investigate how the ability to apply functions in economical and mathematical tasks depends on the two types of knowledge. Finally, a more ambitious aim of the study is to relate the outcome of the analysis to the pedagogical philosophy that has been applied to the study population at the upper secondary school. Despite the attention given the nature and relationships between procedural and conceptual knowledge by researchers in mathematics education, studies where procedural and conceptual knowledge is assessed from large groups of learners seem to be absent. The intention of the present study is to investigate the research questions based on data from a large group of students.

3.1 RESEARCH QUESTIONS

The research questions in this study can be divided into two categories. The first question addresses what we put into the concepts procedural knowledge of functions and conceptual knowledge of functions, while the second and third questions concerns relationships. The intention of this study is to investigate procedural and conceptual knowledge of functions through a large sample and to reflect on the outcome of the analysis from an empirical perspective. Even if researchers in mathematics education have theoretical considerations of the nature of knowledge, one needs to develop measures in order to study their relationships. Haapasalo and Kadijevich (2000) emphasize that procedural and conceptual knowledge cannot be measured directly, but only through procedural and conceptual tasks. One aim of the study is to develop tasks that are reliable and valid measures for procedural and conceptual knowledge of functions. The measures will provide a basis to analyse the relationship between procedural and conceptual knowledge of functions and how they predict the ability to apply functions.

When studying procedural and conceptual phenomena, one must take into account that these phenomena are content dependent (Haapasalo & Kadijevich, 2000). Consequently, a study should restrict the attention towards a specific mathematical concept like functions, and the phenomena that are studied must be characterised for the specific content that is subject to analysis. The following characterisation is used for procedural knowledge of functions, conceptual knowledge of functions and the ability to apply functions in this dissertation:

Procedural knowledge of functions denotes a dynamic and successful use of specific rules, algorithms or procedures when they are applied on functions. This involves a successful use of algorithms step by step and the utilization of rules within different representations separately, such as algebraic and graphic representations including the use of the format and syntax required for the representational system(s) expressing the functions.

Conceptual knowledge of functions denotes a successful utilization of particular networks and relationships related to functions. This includes the utilization of relationships between different representation forms, relationships to other mathematical topics and previous mathematical knowledge. It also includes the ability to choose between appropriate methods and reflect on the outcome of a mathematical task. Conceptual knowledge of functions also includes the ability to think of a function as a unit free from procedures and the possession of control mechanisms to check whether properties are violated or preserved when a solution is evaluated.

The ability to apply functions is characterised as an ability to recognize and solve problems of economics where functions are involved and to choose between appropriate methods. The ability to apply functions also includes the ability to solve mathematical problems involving more advanced mathematical concepts that are built on the concept of functions.

The research questions are:

1 How can procedural and conceptual knowledge of functions be measured?
2 How do procedural and conceptual knowledge of functions relate to each other?
3 How does the ability to apply functions relate to procedural and conceptual knowledge of functions?

Each question is analysed separately, but it is important to emphasize that they are not seen as independent of each other, but as a whole. In fact the possibility of developing one large model that allows the testing of all questions in one analysis makes it possible to ask whether the model provides sound measurements and good estimates for relations. In this regard it is also an aim to test whether the model as a whole is good, regarding the test of the complete model as a synthesis of all the research questions.

If we look at the research questions separately, many similar issues, especially those which concern relational ones, have been addressed earlier by other researchers in mathematics education but are not based on large samples. The analysis of measurement problems like in questions 2 and 3 seems to be given less attention in studies.

Question 1 is related to measurement problems and the questionnaire items in terms of tasks are first of all developed to meet the characteristics of procedural and conceptual knowledge of functions. It is important that the tasks are measuring what they intend to measure, and also that the tasks satisfy certain criteria for consistency. Traditionally students are assessed with respect to skills or procedural knowledge, and consequently questions used in traditional mathematical assessments can be used for measuring purposes. It is more challenging to measure conceptual knowledge, but some studies have provided quantitative tools for this (O'Callaghan, 1994; Byrnes and Wasik, 1991; Baker & Czarnocha, 2002; Kadijevich, 1999). The study by Baker & Czarnocha (2002), analyzing the relationship between an individual's ability to apply procedural knowledge, meta-cognitive reflection and conceptual thought, used students' scores on written tasks through the semester to measure conceptual knowledge. In measuring of procedural knowledge they used the students' course average. As far as quantitative measures for conceptual knowledge are concerned, it seems that the method has been to provide a set of tasks where relations, properties such as order issues and non-quantitative elements play a part. Byrnes and Wasik (1991) used questions about picture-symbol and word-symbol isomorphism as well as questions on order to quantify conceptual knowledge. Kadijevic (1999) claims that object-based thinking can be effectively assessed with objects that are not quantified, typically asked in form of textual questions related to properties, but also such as argumentative questions on true or false claims. Analysis of the answers to such questions will depend on interpretations and consequently give less accurate quantitative scores. One could speak about less reliability.

Research questions 2 and 3 are concerned with the linking of different knowledge types. Numerous researchers have been occupied with these questions, as commented in the previous chapters, but the methodological approach is mostly different from the one in the present study. It is important to note that questions 2 and 3 use the term "relate to". That also involves the investigation of whether there exist causal directions and makes it possible to study whether the particular views suggested by Haapasalo and Kadijevich (2000) can be supported.

Question 2 addresses how procedural and conceptual knowledge of functions relate to each other. The statistical model does not prove causality, but the task performances will indicate which of the causal directions indicated by the arrows in the figures below are supported. Based on Sfard's (1991) theory of development from operational to structural understanding through operation, condensation and reification one could assume that the causal direction corresponds to Figure 3-1. The view (Haapasalo & Kadijevich, 2000) is supported by this causal direction is the Genetic View.

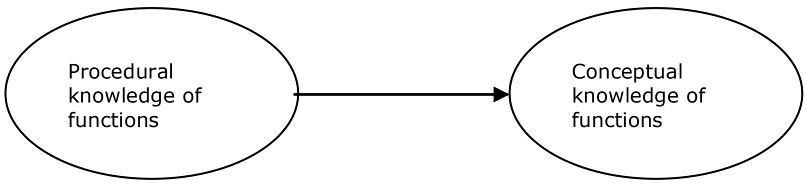


Figure 3-1. Possible outcome supported by the genetic view.

Another possible outcome may be that conceptual knowledge of functions is a necessary condition for procedural knowledge of functions. This view, the dynamic interaction view examined by Byrnes and Wasik (1991), is shown in Figure 3-2.

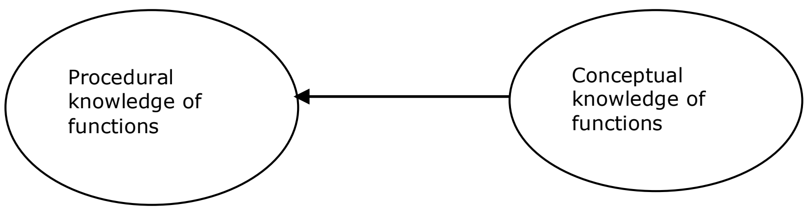


Figure 3-2. Possible outcome supported by the dynamic interaction view.

The simultaneous activation view assumes that conceptual knowledge is a necessary but not sufficient condition for procedural knowledge, as recognised by Byrne (1998) and Haapasalo (1993). In this case, the causality is bidirectional as indicated in Figure 3-3.

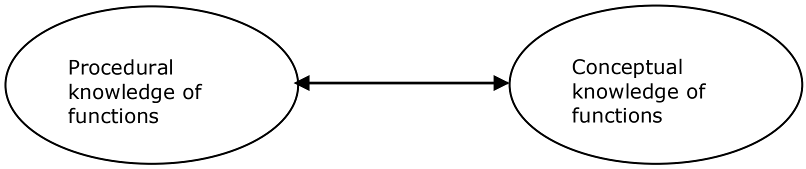


Figure 3-3. Possible outcome supported by the simultaneous activation view.

Understanding a mathematical problem is not only seen as a goal in itself, but as facilitating the achievement of other goals (Schoenfeld, 1982) as being able to apply mathematical knowledge in other fields.

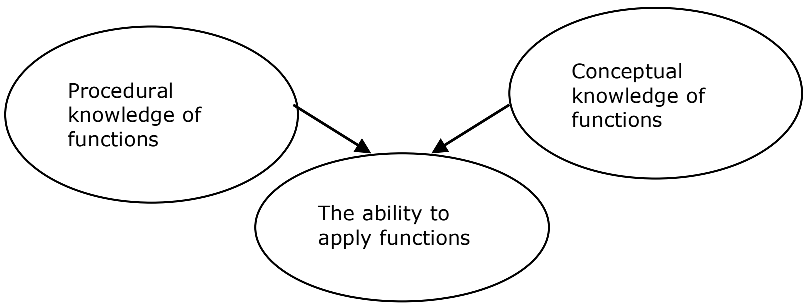


Figure 3-4. The ability to apply functions depending on procedural and conceptual knowledge of functions when intermediate effects between the two explanatory variables are disregarded.

The last research question is also assuming a causal relationship by asking how the ability to apply functions depends on the two knowledge types. As stated earlier, the model integrates the separate hypotheses into one model as shown in the path diagram in Figure 3-4. The figure covers all the aspects that will be tested and discussed in relation to the second and third research question.

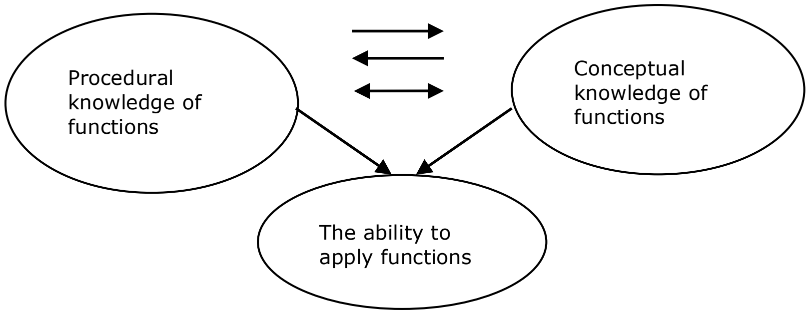


Figure 3-5. Possible relationships investigated in the study.

The main part of the study aims at studying the research questions within the framework of one statistical model, consisting of a measurement part and a structural part. The measurement part of the model will be used as evidence for the answers to research question 1, while the structural part addresses research questions 2 and 3. The intention is to conclude on a final structure by deciding which of the arrows in Figure 3-5 that are supported by the data and which are redundant.

A more ambitious aim of the study is, through interviews, to find supplementary interpretations of the quantitative outcome and shape a perspective of discovering new ideas or interesting questions and hypothesis for further research. The aim is not to generalize findings but rather to understand and reflect on the outcome of the statistical research and such aspects that are not covered by the research questions. The interview addresses students' beliefs of mathematics, beliefs of themselves as learners of mathematics, and beliefs of their educational experiences from upper secondary school.

4 Methods

The subjects of this study were studying mathematics as a part of their studies in economics at a business school, and the data of this study was collected from students in the first year of the study. Tasks have developed to find out how the students perform on procedural tasks, and to what extent they reveal procedural knowledge. In addition, the tasks have been planned to measure the ability to apply mathematical concepts in economics and differentiation. The tasks were developed to satisfy explicit criteria covering the essence of the variables measuring knowledge. All the variables are included into one psychometric model. The model as a whole includes several hypotheses on how to measure knowledge and ability to apply functions. It also reflects on how these concepts relate or depend on each other.

To carry out the test, knowledge of functions must be made operational. More specifically, it requires characterisation of the items 'procedural knowledge of functions', 'conceptual knowledge of functions' and the 'ability to apply functions'. A set of tasks is provided to measure these items by using a confirmatory factor analysis. This part of the study is referred to as the measurement part. For an evaluation of an analysis to be meaningful, it is obvious that the variables included measure what they intend to measure. In other words, a set of tasks that are meant to reflect a student's procedural knowledge of functions, should ideally measure procedural knowledge of functions and nothing else. In order to be able to do that, procedural knowledge of functions has to be decomposed into suitable criteria that are measurable and correspond to the generally accepted meaning of the term. Even if there is a consensus among researchers and teachers on the meaning of procedural knowledge, the situation becomes more complex when it comes to conceptual knowledge. The ability to apply functions, which is not an established term, is measured by tasks addressing the ability to apply functions in economic applications as well as a "building block" in mathematics at a more advanced level.

It might seem difficult or even impossible to develop algorithmic skills without having some kind of conceptual knowledge, at least on the more basic components involved in the operations, and it is unlikely that a student can develop skills without being able to reflect on the results. On the other hand, understanding a concept regardless of the underlying procedures seems almost impossible, at least for the non-mathematician. Several theories have discussed how different knowledge types depend on each other. The rationale for the initial hypothesis about relationships relies on the earlier research by Sfard (1991), Hiebert & Lefevre (1986), Kadijevich and Haapasalo (2001) and others. This part of the model is referred to as the *latent variable model.*

The *measurement model* and the latent variable model are combined in a *structural equation model* implemented in the software package LISREL (K. Jöreskog & Sörbom, 1993). Appendix B outlines in general the theory of structural equation modelling that is applied in this study. These types of models are widely used in the social and human sciences, especially when studying phenomena where variables cannot be measured directly, as often is the case with mathematical understanding. Even if causal relations cannot be proven, the analysis will tell us whether the relations suggested in the model match the sample of data (Bollen, 1989).

All the research questions raised in this study are formulated as hypotheses which are embedded in one statistical model, but the model is not only a tool for testing hypotheses, but also a tool for the discussion of such issues like students approaches to learning, teaching practices, use of assessments and use of computer environments and calculators.

This chapter presents the model, which is hypothesized on the basis of the theoretical considerations and the research questions that were discussed in chapter 3. Following the general model theory, the background for the specific model applied in the dissertation, as well as a discussion of the statistical methods that are involved, is described. Each concept and the items that are supposed to measure each concept are outlined. Different estimation techniques for estimation of parameters as well as different fit measures are described. Chapter 4.6 describes the measurement model with special focus on confirmatory factor analysis. Chapter 4.7 gives a background for the latent variable model, before the synthesis of the two parts of the model is completed in chapter 4.8.

4.1 DESIGN OF THE STUDY

To meet these challenges a design in four stages was applied, denoted as the pilot test, the main test, the post test and the interview respectively.

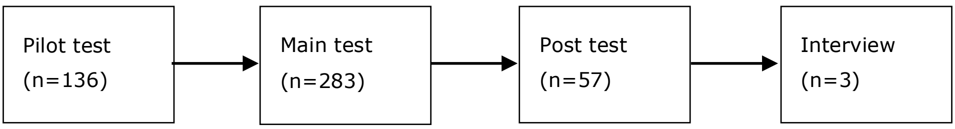


Figure 4-1. Stages of the study.

The three tests were given to different groups of students at different times, but the students were all first year students taking the same course in mathematics. In addition, three students were interviewed about their mathematical beliefs and educational background from upper secondary school to find possible explanations for the outcome of the main test.

The intention of the pilot test is to try out different types of tasks in an explorative manner to have some background for developing the tasks for the main test. The results are used to reflect on the type of knowledge required, the level of difficulty and the tasks' ability to reveal distinctions between students.

The *main test* is intended to develop the statistical model and perform analyses to study the research questions. The development of the model includes an estimation of the model parameters, an evaluation of model fit and the possible justifications of the model. One of the challenges in this process was to develop an appropriate set of tasks in order to measure the different types of knowledge. Another challenge was to find a suitable method to judge whether the measures in the model are valid in the sense that they measure what they intend to measure. To meet the challenge of studying concepts like conceptual knowledge of functions, which is vague and difficult to measure directly, these concepts were treated as factors using a factor analysis technique. Confirmatory factor analysis was applied to develop tasks to measure three concepts: procedural knowledge of functions, conceptual knowledge of functions and the ability to apply functions. Relationships between the concepts were investigated by linear regressions. A structural equation modelling technique allowed integrating the factor analysis and the regression analysis into one statistical model.

The purpose of the *post test* is to provide a tool to evaluate the validity of the measures by using a dataset different from the dataset used in the main test. Since validity is hard to measure by applying the data from the main test alone, the same questions that were used in the main test were given to a new group of students. The results from the test were compared with the students' performance in their exams in mathematics and business economics. The idea is that if a measure of for example procedural knowledge of functions is valid, it is likely to predict the students' performance in other skill-oriented mathematical tasks about functions. The validity of the measure for conceptual knowledge of functions is evaluated in a similar way. Finally, the validity of the measure for the ability to apply functions is held up against the students' performances at their exams in business economics. The post test was also designed to evaluate the relations between the different types of knowledge that were found in the main test using new data and a different test design.

4.2 STRUCTURAL EQUATION MODELING

As suggested by the research questions, we have to deal with concepts that need to be measured in some way in order to discuss how they are related. *Procedural knowledge of functions*, *conceptual knowledge of functions* and *the ability to apply functions* are such concepts and will be represented by variables. The problem with such variables, in the same way as for example variables that represent intelligence or depression, is that they are often vague and almost impossible to measure directly in a simple way. Instead we need a method to indirectly measure such concepts through observable items. An appropriate methodology is to use structural equation modelling, which is a multivariate statistical technique combining multiple regression and factor analysis.

Structural equation modelling has become a widely used statistical technique in the social sciences, especially in psychology. Although the use of these models has evolved rapidly the last decades, the development started early in the nineteenth century. Spearmann (1904) discussed the concept of general intelligence by observing correlations between performances on several problem-solving tasks. He made the observation that children's scores on different performance tests, which did not seem to be connected, were correlated. He assumed that this was caused by a common underlying factor, general intelligence. This way of treating a concept, as a construct measured through several observable variables, is the main idea in factor analysis and is therefore suitable for the present study. In structural equation models, the factors are represented by latent variables that cannot be measured directly. Thurstone (1947) was also important in the early development of factor analysis for psychometric research (Steiger & Schönemann, 1978, p. 171), as he emphasizes that it was not the individual factor scores that were essential, but rather the discovery and nature of the factors themselves. It is therefore important for the model in this analysis to identify the different aspects of the three factors.

Today, two designs of factor analysis are common. One is exploratory factor analysis used to discover underlying factors from a set of variables. In other words, the intention of the analysis is to discover common factors for a group of variables in order to treat them as one factor. The other design, which is used in this study, is confirmatory factor analysis where a priori assumptions on the link between factors and variables are tested. The assumption on the structure between factors and variables is based on previous research in mathematics education and on judgment. This analysis is designed to see whether the collected data supports this structure. The word variable is here used to address observable variables. Later, these variables are also referred to as items, while the factors themselves are denoted latent variables.

Wright (1934), a biologist, invented path diagrams to represent linear relationships between variables. Path analysis has been used to explore relationships between attitude and mathematical performance, when seeking to study how such things as self-efficacy and self-concept beliefs contribute to mathematical problem solving (Pajares & Miller, 1994). The path diagrams are essential in structural equation modelling, and are used to describe causal relation between variables by use of arrows. Wright proposed rules to relate the correlations or covariances of variables to equations. Relationships between procedural knowledge of functions, conceptual knowledge of functions and the ability to apply functions are represented by path diagrams to illustrate relationships.

According to Bollen (1989), three components are present in structural equation modelling: path analysis, conceptual synthesis of latent variables with measurement models and finally general estimation procedures. Factor analysis, latent variables, path diagrams and equations are key components in a structural equation system, and it is the synthesis of these components that constitutes the model theory. In this study, the observations are collected from the students' outcomes in tests. Three items, 'procedural knowledge of functions', 'conceptual knowledge of functions' and the 'ability to apply functions', constitute the factors in this study. As mentioned earlier, the word 'concept' is used often in general terms, not in the meaning 'mathematical concept'. As an example, 'function' is not what is meant in this context, while 'conceptual knowledge of functions' is. Multiple regression equations are used to study dependencies between the latent variables, treating them as dependent or independent variables. In fact each variable can occur in different equations simultaneously. Some concepts may appear to be both dependent and independent variables, as conceptual knowledge of functions might depend on procedural knowledge of functions, but it may also be an explanatory variable for the ability to apply functions. Problems with such endogenous variables are difficult to treat within normal linear regression methods, but are handled properly when a structural equation modelling technique is applied.

Even if it is usual to distinguish between the measurement part of the model and the structural part of the model, the factor analysis and the regression equations altogether is said to constitute a structural equation model. To distinguish between the two components of the model, the terms measurement model and latent variable model will be used. Since the intention is to explore a theory on mathematical understanding, some knowledge on the underlying latent variable structure commented upon earlier and hypothesized in the research questions is assumed. In contrast to exploratory factor analysis where the links between the observable and latent variables are uncertain, confirmatory factor analysis seems appropriate in this situation where we postulate relations a priori (Byrne, 1998).

If we let the items be scored by task-scores and we let the latent variables represent different kinds of mathematical knowledge, then we have a model structure that is suitable to study relationships between different types of knowledge. The idea is to set up a model based on assumptions founded in theory and practice, estimate the parameters and find out if the collected data supports the model. In general, procedural knowledge of functions is probably easier to measure than the more complex and vague concept of 'conceptual knowledge of functions'. The question is not whether a student has achieved a conceptual knowledge of functions or not, but to what extent he or she has accomplished different aspects of such knowledge. A "good" model should account for the different facets of knowledge, as well as the degree to which each criterion is met. It is important to notice that the structural equation model analysis does not serve as proof for causality between the variables. When variables are measured at the same point in time, as in this study, it is not possible to draw conclusions on causal relations between them (Cramer, 2003, p. 91). However, the measurement part of the model will provide a possibility to estimate scores on procedural and conceptual knowledge of functions for each student that can be used to investigate causality.

The notation is in accordance with the standard LISREL notation and the model is described in a complete path diagram as well as by equations and matrixes. Three concepts are represented by three latent variables in the model, and are tested through a set of tasks given to the students. For this study, three concepts (latent variables) are included in the model:

Procedural knowledge of functions (ξ_1)
Conceptual knowledge of functions (η_1)
Ability to apply functions (η_2).

4.3 PROCEDURAL KNOWLEDGE OF FUNCTIONS

The questions aimed at testing procedural knowledge of functions are typically questions where the students are asked to calculate values for a given function for different values of the arguments, drawing functions or solving inequalities. Typically, these questions are solved following a step-by-step procedure without a need for an in-depth knowledge of functions.

Procedural knowledge of functions was measured by the following variables (items):

Graphic procedures (x_1)
Algebraic procedures (x_2).

'Graphic procedures' refer to actions as for example drawing a graph or reading from a graph. Such operations must be distinguished from operations where the graphs are treated as units for the operation, as for example when two functions are added by adding their graphs. In the same way, 'algebraic procedures' refer to problems where one operates on one function such as computing a value.

Even if other representation forms, such as tables and texts, are used in mathematics, it is hard to create tasks for these types of representations that distinguish between the use of procedural and conceptual knowledge, especially for texts. A textual representation of problems and their solutions probably requires more conscious thinking, and is possibly more related to conceptual knowledge. Since graphs and algebraic expressions are the dominant representation forms in teaching and learning of mathematics, and also are clearly distinct from each other, the measurement of procedural knowledge of functions is limited to these two variables.

4.4 CONCEPTUAL KNOWLEDGE OF FUNCTIONS

When it comes to measuring *conceptual knowledge of functions*, we need characteristics for objectification, and these characteristics needs to be measurable or at least to some extent observable. One aspect of having a conceptual knowledge is to have a good sense of isomorphism between different symbolic representations, keeping the same object in mind. The students in this study were asked questions where they had to shift between different representation forms. One example of this is to point out which graph that corresponds to a given algebraic expression.

Another type of task to measure conceptual knowledge of functions consists of questions where students are asked to perform operations on graphs where only the graphs are presented and the corresponding algebraic expressions have been left out. The intention is to test the students' ability to operate on graphs as units, without going into any procedural steps on the given functions.

A third set of tasks asks the students to give interpretations of functions given by algebraic expressions. These tasks are designed to measure the students' ability to make reflections on the given functions, rather than to perform algorithmic procedures.

Conceptual knowledge of functions was measured by the following variables (items):

Relations between graphic and algebraic representations (y_1)
Graphic interpretations (y_2)
Algebraic interpretations (y_3).

The first item addresses the ability to see relations between different representations of functions. In the second item, it is the graph itself that is treated as one object. In other words, the graph is a "unit" which is treated as a whole. This is different from the item "algebraic procedures" (x_2), where operations are performed on the elements within the graph, such as reading a value from the graph. In this item, also the term 'nonprocedural' could be used, referring to an absence of algebraic procedures. In general, one might say that a graph is a more condensed representation of a function than an algebraic expression, and that the algebraic expression carries more detailed information on how to carry out procedures. The idea is to ask questions where the graph carries sufficient information to solve the task, without knowing details on procedures.

The third item is related to how functions can be treated as units when they are represented only by names like *f,* and some properties are given as a text. The functions are used as entities in the sense that, for example, one must decide what kind of properties the product of two functions has, given some properties of the two functions involved. One can say that students are asked to interpret the meaning of symbols like $f(x)\cdot g(x)$. The meaning of a symbol arises from the connection between the symbol and the object (function) to which it refers (Edwards, 1998, p. 70). If we think of symbols like $f(x)$ as carriers of the meaning of a function, the third item challenges the student to extract meaning.

4.5 ABILITY TO APPLY FUNCTIONS

The *ability to apply functions* can be thought of in two ways. One way is to apply the concept of function on a mathematical concept to operate on a more advanced mathematical concept at a higher level, such as differentiation or integration. Another is to look at applications in another subject than mathematics, such as economics or statistics. Since the students are at their first year of their study in economics, it is natural to use examples from this domain. The economics examples are simple, typically related to cost and income situations.

The tasks in derivation are of two types, where one is the traditional derivation of functions given algebraically. The other type is a group of tasks to reveal how derivation is understood by graphic representations. The ability to apply functions was measured by the following variables:

Economic applications (y_4)
Derivation (y_5)
Graphic knowledge of a function and its derivative (y_6).

The item that measures economic applications is related to problems involving economic phenomena such as costs, income, profit and also the relation between them. The tasks within the item relate to a reality outside the world of mathematics, but where mathematical knowledge is supposed to be relevant.

There is no doubt that derivation, as measured by the second item, is in one regard a procedural task, and a relevant question is whether it measures conceptual knowledge or not. However, the operations refer to a more advanced concept than functions themselves. Another aspect is the ability to reflect on the answers as for example being aware that the exponent of the derivative f' of a polynomial function f should be one degree lower than that of the original function.

The last item concerns relational knowledge, asking the student to relate the graph of a function with the graph of its derivative. In many ways, this item points to the core of the concerns that have motivated for this study, namely that many students seem unable to interpret derivation from graphs.

4.6 THE MEASUREMENT MODEL

The research questions and the discussion so far are reflected in the path diagram in Figure 4-2 for the measurement model. The measurement model and the latent variable model, as described in the previous chapters, form the basis for developing a structural equation model in LISREL. It addresses many issues, not only regarding estimation methods and fit measures, validity and reliability, but also on statistical assumptions. The suggested structure that is visualized in the path diagrams is a suitable model for the statistical study of the research questions.

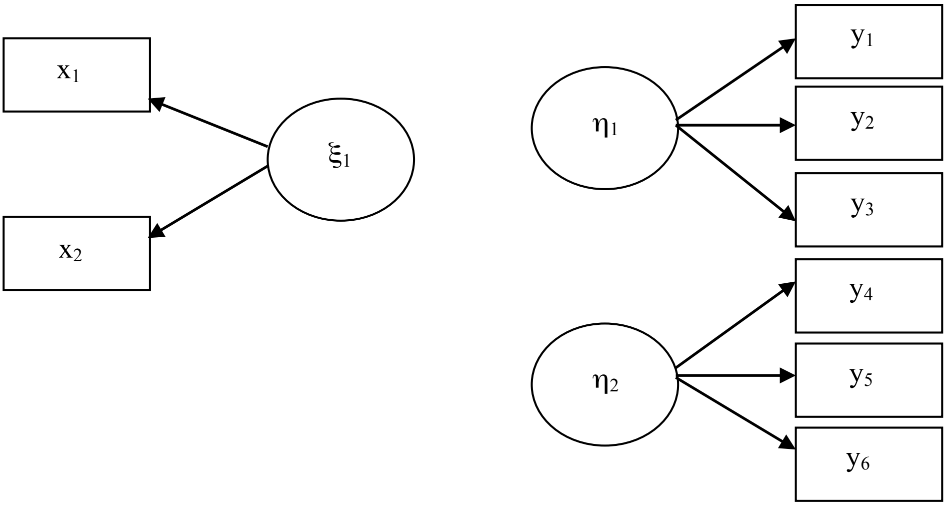


Figure 4-2. The hypothesized measurement model with three latent variables and eight items.

In LISREL exogenous latent variables are represented by ξ's, and endogenous latent variables by η's. In this notation, ξ_1 is the name of the latent variable "procedural knowledge of functions", η_1 denotes "conceptual knowledge of functions" and η_2 "Ability to apply functions". In the next chapter, outlining the complete model, the two latter are endogenous[5]. The direction of the arrows indicates a causal direction, often called a reflective (as opposed to formative) model, which means that the ability to score on the items is considered to be a consequence of the latent variable.

The measurement model can also be described by the following equations:

$x_1 = \lambda_1\xi_1 + \delta_1$ (4.1)
$x_2 = \lambda_2\xi_1 + \delta_2$ (4.2)
$y_1 = \lambda_3\eta_1 + \varepsilon_1$ (4.3)
$y_2 = \lambda_4\eta_1 + \varepsilon_2$ (4.4)
$y_3 = \lambda_5\eta_1 + \varepsilon_3$ (4.5)
$y_4 = \lambda_6\eta_2 + \varepsilon_4$ (4.6)
$y_5 = \lambda_7\eta_2 + \varepsilon_5$ (4.7)
$y_6 = \lambda_8\eta_2 + \varepsilon_6$ (4.8)

which in matrix notation is:

$$\mathbf{x} = \mathbf{\Lambda}_x\boldsymbol{\xi} + \boldsymbol{\delta} \quad (4.9)$$

$$\mathbf{y} = \mathbf{\Lambda}_y\boldsymbol{\eta} + \boldsymbol{\varepsilon} \quad (4.10)$$

where:

[5] In general, it is possible for two factors to load on the same item.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} \quad \mathbf{\Lambda_x} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \quad \mathbf{\Lambda_y} = \begin{bmatrix} \lambda_1 & 0 \\ \lambda_2 & 0 \\ \lambda_3 & 0 \\ 0 & \lambda_4 \\ 0 & \lambda_5 \\ 0 & \lambda_6 \end{bmatrix}$$

$$\boldsymbol{\eta} = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \quad \boldsymbol{\xi} = \xi_1 \quad \boldsymbol{\delta} = \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} \quad \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix}$$

The δ's and ε's, are the errors of measurement having an expected value of zero and are assumed to be uncorrelated with the ξ's, η's and each other. They are also expected to be homoscedastic and non-autocorrelated.

4.7 THE LATENT VARIABLE MODEL

In the other part of the model, referred to as the latent variable model, the concepts (latent variables) are treated as variables in a set of simultaneous linear equations where each latent variable can be considered both as a dependent or independent regression variable.

Since we want to explore a theory on mathematical knowledge as given by the research questions, we assume some knowledge on the underlying dependencies between procedural knowledge of functions and conceptual knowledge of functions.

The first assumption is that the ability to apply functions is a function of the two other latent variables. At the initial stage, conceptual knowledge of functions is assumed to be dependent of procedural knowledge of functions[6]. This assumption is in fact to assume the genetic view as illustrated in Figure 3-1, and might seem strange. However, this will be used to estimate the parameters in the measurement model, which is used to produce factor scores and re-evaluate the assumption on the causal direction. It is important to notice that these estimates are independent of the assumed causal direction.

In fact, procedural knowledge of functions (ξ_1) is assumed to have both a direct and an indirect effect on the ability to apply functions (η_2). The indirect effect goes via conceptual knowledge of functions (η_1). In other words, the total effect can be decomposed into a direct and an indirect effect.

Total effect = Direct effect + Indirect effect

The separation of effects plays an important part when it comes to the interpretation of the results. If the genetic view is supported, a large direct effect will indicate that it is possible to apply functions even if the conceptual knowledge of functions is low. If the indirect effect is the dominant one, then procedural knowledge alone does not seem to be

[6] One could argue that η_1 is a condition for ξ_1, in other words that we have a bidirectional effect, but this would have no impact of the model fit.

sufficient to be applied at a high level. If the re-evaluation should support the dynamic interaction view as in Figure 3-2, then conceptual knowledge of functions will have an indirect effect on the ability to apply functions via procedural knowledge of functions, in addition to a direct effect. So, initially, the hypothesized structure is the structure, shown in Figures 3-1 and 4-3.

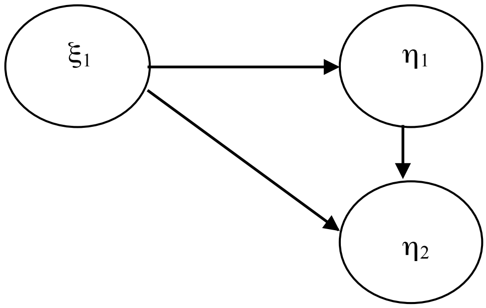


Figure 4-3. The initially hypothesized latent variable model.

It is also important to interpret the effect that procedural knowledge of functions (ξ_1) has on conceptual knowledge of functions (η_1). This effect will indicate whether a certain level of procedural knowledge of functions (ξ_1) is required to achieve conceptual knowledge of functions (η_1). Since conceptual knowledge of functions is treated as an endogenous variable, explained only by procedural knowledge of functions, it is implicit in the hypothesized model that we expect this effect to be significant. If not, it might be expected that other explanatory variables should have been included.

Finally, there is also a hypothesized effect from conceptual knowledge of functions (η_1) on the ability to apply functions (η_2). It seems obvious that this effect is expected to be significant. The structural model diagram reflects hypotheses about the relations between the latent variables. The direction of the array from procedural to conceptual knowledge assumes support for the view that procedural knowledge of functions is a condition for conceptual knowledge of functions, and precedes conceptual knowledge in time. As stated earlier, this must not be interpreted as if the direction of causality is known beforehand. We could have hypothesized an opposite or bidirectional causality.

The corresponding equation form is:

$$\eta_1 = \gamma_{11}\xi_1 + \zeta_1 \qquad (4.11)$$

$$\eta_2 = \beta_{21}\eta_1 + \gamma_{21}\xi_1 + \zeta_2 \qquad (4.12)$$

which in matrix format is described by:

$$\eta = B\eta + \Gamma\xi + \zeta \qquad (4.13)$$

where:

$$\mathbf{B} = \begin{bmatrix} 0 & 0 \\ \beta_{21} & 0 \end{bmatrix} \quad \mathbf{\Gamma} = \begin{bmatrix} \gamma_{11} & 0 \\ \gamma_{21} & 0 \end{bmatrix} \quad \xi = \begin{bmatrix} \xi_1 \\ 0 \end{bmatrix} \quad \boldsymbol{\zeta} = \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}$$

where γ's and β's are regression parameters and ζ's are disturbance terms.

4.8 THE COMPLETE MODEL

Combining the measurement model and the latent variable model discussed in the previous chapters, gives the path diagram in Figure 4-4 for the *complete model*. The model meets the criterion for over-identification (Byrne, 1998, p. 29) since the number of estimable parameters is less than the number of data-points. We have 8 λ's, 2 γ's, 1 β, 2 δ's and 6 ε's, altogether 19 parameters to be estimated. In general the number of data-points, being the number of covariances and variances (or respectively correlations and 1's) is calculated as $p^*(p+1)/2$ where p is the number of observable variables. Since we have 8 observable variables, the number of data-points is 36. Thus, with 36 data-points and 19 parameters to be estimated, we have an over-identified model with 17 degrees of freedom. A positive number of degrees of freedom is necessary so that we are able to eventually reject the hypothesis that the model fits the data, and is therefore required for the analysis to be of any interest.

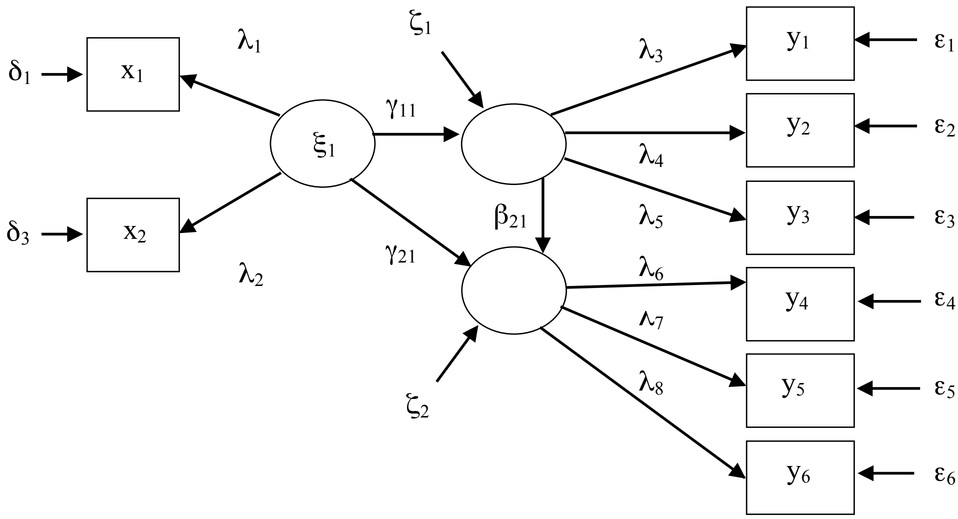


Figure 4-4. The complete structural equation model with parameters. The regression parameters (γ_{11}, γ_{21}, β_{21}), the factor loadings (λ_I's) and the disturbance terms (δ_I's and ε_I's) are estimated.

Conceptual knowledge of functions (η_1) and ability to apply functions (η_2) are endogenous, but an endogenous latent variable is probably determined partly by the model. ζ_1 and ζ_2 are the random disturbance terms being undetermined (Bollen, 1989, p. 12)[7].

It is important to be aware of the different kinds of effects the variables have on each other to achieve a better understanding of the path diagram. Even if causality cannot be proven, the arrows in combination with the estimated parameters indicate the effect of one variable on another. A one-unit change in ξ_1 leads to λ_2 changes in x_2 and λ_1 change in x_1. Similarly, a one-unit change in η_1 leads to β_{21} changes in η_2. These effects are called direct effects, as they are not mediated by another variable in the path diagram.

[7] ζ_1 and ζ_2 are not included as parameters to be estimated in the calculation of degrees of freedom.

However, a change in ξ_1 will also lead to changes in for example y_2 and η_2, which are intermediated by η_1. The effect of ξ_1 on y_2 is $\gamma_{11}\lambda_4$, meaning that one unit change in ξ_1 will lead to $\gamma_{11}\lambda_4$ units of change in changes in y_2. One unit change in ξ_1 will lead to both γ_{21} change in η_2 in addition to a $\gamma_{11}\beta_{21}$ change in η_2. This last contribution is referred to as the indirect effect. One variable's effect on another variable is the sum of the direct and indirect effects (Bollen, 1989, p. 36). The total effect on η_2 from ξ_1 is therefore expressed mathematically as $\gamma_{21}+\gamma_{11}\beta_{21}$.

The analysis in this study is based on the correlation matrix rather than the covariance matrix. This has some effects on the interpretation of the estimated parameters as they measure the expected change in the dependent variable in standard deviation units caused by a one standard deviation change in the independent variable. In this way the estimates are independent of scale and effects from different variables can be compared directely.

Research questions 1 to 3 are all studied within the frame of this model or a similar model modified for causal directions and redundant relationsships. In fact, each arrow in the path diagram represent an hypothesis. The research question 1 will be investigated through the measurement part of the model, while questions 2 and 3 are subject to the latent variable part of the model.

Based on the observed covariance matrix S, between the latent variables $x_1,x_2,y_1,y_2,y_3,y_4,y_5$ and y_6, the factor loadings and regression parameters will be estimated. The first research question is:

Research question 1: How can procedural and conceptual knowledge of functions be measured?

This question can be studied from two perspectives. One is the intepretation of the magnitude of the parameters, while the other is to investigate whether the items constitute a question battery which is valid and reliable. As already mentioned, a standardized solution will allow one to compare the relative effects from the different items directely. As an example, one can study which category of questions (item) that seems to have most effect on conceptual knowledge of functions.

Research questions 2 and 3 can be investigated within a paradigm of traditional testing of hypothesis. The models allow us to determine whether these effects are statistically significant or not. Research question 2 is in fact tested as a simple linear regresssion:

Research question 2: How do procedural and conceptual knowledge of functions relate to each other?

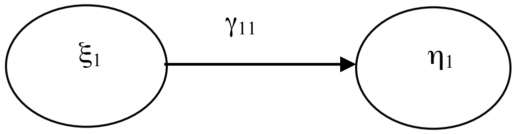


Figure 4-5. The relationship relevant for research question 3, indicated with an arrow.

The question is whether the parameter γ_{11} in equation (4-14) is significant:

$$\eta_1 = \gamma_{11}\xi_1 + \zeta_1 \qquad (4.14)$$

Looking at equation (4.14), research question 2 adrress the bfollowing hypothesis[8]:

$H_{0;2}$: $\gamma_{11} = 0$

$H_{1;2}$: $\gamma_{11} > 0$

The alternative hypothesis is one-sided and is tested by a t-test at 5 % level of significance. There is little reason to test this two-sidedly, as there is no reason to assume that increased performance in procedural knowledge should have a negative effect on conceptual knowledge in general for a large group of students.

The last research question is, as mentioned earlier in this chapter, concerned with the requirements for the ability to apply functions.

Research question 3: How does the ability to apply functions relate to procedural and conceptual knowledge of functions?

The question can be separated into two parts. First we regard the direct effects from procedural and conceptual knowledge of functions to the ability to apply function as illustrated in Figure 4-6.

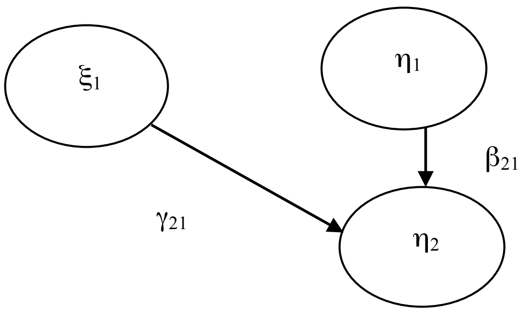


Figure 4-6. The path diagram of direct effects: from procedural knowledge of functions (ξ_1) and conceptual knowledge of functions (η_1) to the ability to apply functions (η_2).

The second part is to regard conceptual knowledge of functions as an intermediate effect as in Figure 4-7 by combining the direct and indirect effects.

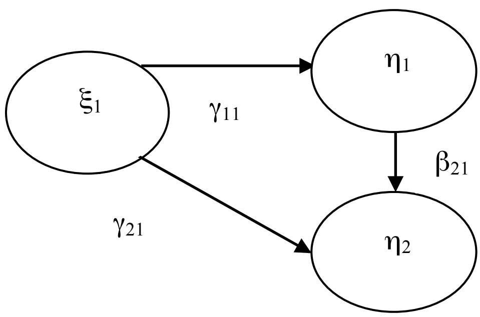


Figure 4-7. The path diagram of direct and indirect effects: from procedural knowledge of functions (ξ_1) and conceptual knowledge of functions (η_1) to the ability to apply functions (η_2).

[8] The number after the semicolon addresses the research question number.

The relations in Figure 4-6 have to do with equation (4.15):

$$\eta_2 = \beta_{21}\eta_1 + \gamma_{21}\xi_1 + \zeta_2 \qquad (4.15)$$

The parameters β_{21} and γ_{21} are tested separately one-sided by t-tests, while an R-Square value, ranging from 0 to 1 will indicate whether the equation (4.15) as a whole is good.

The first hypothesis[9] related to research question 3 concerns whether an increase in procedural knowledge of functions has a direct effect on the ability to apply functions:

$H_{0;3\text{-}1}$: $\gamma_{21} = 0$
$H_{1;3\text{-}1}$: $\gamma_{21} > 0$

A similar test is done to test the direct effect of conceptual knowledge of functions on the ability to apply functions:

$H_{0;3\text{-}2}$: $\beta_{21} = 0$
$H_{1;3\text{-}2}$: $\beta_{21} > 0$

Both tests are one-sided as it seems very unlikely that increased procedural or conceptual knowledge shouls have a negative effect on the ability to apply functions.

It might be that one could argue for competing models, or that the analysis itself would suggest a model that fits the data better and competing models will be evaluated. The model represented in this chapter is the initially hypothesised model based on the discussion and asumptions, presented so far.

4.9 DATA COLLECTION AND STUDY POPULATION

This study was conducted in Norway at the Norwegian Business School (BI), and data for the main analysis was collected from 283 students studying mathematics as a part of a three-year bachelor's course in economics. Since the development of a suitable set of tasks was considered critical, a pilot study, including 136 students, was done one year before the main trial. The experiences from the pilot study were used to develop suitable tasks to be employed in the main test. A post test was conducted to discuss validity and relational aspects by collecting data from 57 students. Finally three students with educational background from different upper secondary schools in Norway were chosen to form a quasi peer group. They did not participate in doing the whole main test but took the same course in mathematics as the students in the main test. In all, the study is based on data from 479 first-year students at BI.

BI is a business school offering bachelor, master and doctoral programs in business economics and marketing. The school is located at different locations in Norway, with a total of approximately 20,000 students, where most of the students are full-time students in their early twenties.

The participating students' background in mathematics could be described as weak, although there are substantial variations between them. Only the lowest level of mathematics from the 3-year upper secondary school is required for enrolment at the

9 Research question 3 is decomposed into two sub-questions as indicated by the subscripts 3-1 and 3-2.

bachelor study. The students were not asked for their mathematical background in the test, but approximately 60% had 1 year of mathematics at high school while the rest had two or three years. A general impression is that many students struggle with mathematics, and approximately 30-40% normally do not pass the final exam. Some students were part-time students, which means that they use two years to take the courses that fulltime students complete in a year. The duration of the test was set as three hours, and the use of calculator was allowed. The pilot test took place in Oslo with 136 participating students.

After the evaluation of the pilot test, the main test was conducted with data from 283 students from three different locations, Oslo, Sandefjord and Lillestrøm. The number of students was distributed as follows:

Class	Students
Sandefjord part-time	13
Sandefjord full-time	27
Lillestrøm part-time	21
Oslo class 1 full-time	105
Oslo class 2 full-time	117
Total	283

Figure 4-8. Number of subjects per location.

Finally data was collected from 57 students in Trondheim to evaluate validity and the relations between the knowledge types. Since the purpose of the post test was to evaluate the measures that were applied in the main test, the questions in the post test were exactly the same as in the main test.

For practical reasons it is not common to select a purely random sample on individual basis in this kind of study. Instead, a number of classes from different schools are selected, and all students in these classes are included in the study. Such sampling is referred to as cluster sampling (Ary, Jacobs, & Razavieh, 1996) where the classes are clusters. Even if the sample is not purely random, the sample represents the population of this study branch in Norway reasonably well. It is possible to test differences between full-time and part-time students, or between students in larger and smaller classes, using this sample, but that was not the intention with the study. Rather, the intention was to include clusters that were representative of the study population.

The classes were comparable in the sense that all students followed lectures three hours weekly, though with different teachers. To ensure that the test was performed in a similar manner at all locations, I supervised all the tests. In Oslo and Sandefjord, the tests were done in classes immediately after each other to avoid exchange of information between the students in different classes. All answers were written on transparent sheets in three copies, of which the student kept one. Another copy was later returned to the students with corrections and comments to ensure that the students put effort in their work with the test. No mark was given, as the test was not a part of the official evaluation of the course, and the students who joined the test were informed about this. The use of books and calculators was allowed to make the test situation resemble familiar the exam situations the students already know.

The pilot study served as a tool to evaluate different types of tasks to develop suitable tasks for the main test. Therefore the results from the pilot study were not analysed with the hypothesised model. Instead a combination of judgement and simple statistics was used to evaluate the tasks with respect to their ability to measure different aspects of knowledge. The experiences from the pilot study are presented in the following chapter where the development of the tasks in the main test is described.

The interviews were semi-structured clinical interviews with a combination of planned questions, but letting the students speak for themselves. The questions were written in an interview guide and were designed to focus on the aim of the interview. The interview questions were planned from two perspectives. The first perspective was to ask the students about beliefs of mathematics while the second perspective was to find out more about the students' educational history. The students were interviewed separately in an interview room for about 30-40 minutes each.

4.10 TASKS MEASURING THE ITEMS

This chapter gives a description of the tasks in the main test for each item, and the related latent variables. Since the validity of the tasks can be judged by the content mainly, the discussion is quite detailed. One often thinks of a "task" as skill-oriented, whereas the word "problem" is associated with something that involves more than just procedures. In the present chapter, the word "task" is used when describing the different items, whether they measure procedural knowledge of functions, conceptual knowledge of functions or the ability to apply functions.

It is important to keep in mind the different challenges that are required to solve a mathematical problem. One may refer to Polya's model of problem solving processes which consists of four steps (Polya, 1945):

1 Understanding the statement of the problem
2 Conceiving a plan for its solution
3 Executing the plan
4 Verifying and assessing the answer

These four steps can be helpful references when the criteria for the different tasks are discussed. The relevant question is: What kind of knowledge is required by each of the four steps? At first sight, it seems that stage three is the "procedural" stage, while it is hard to see how procedures alone are sufficient to work through the other stages. All stages are relevant with respect to conceptual knowledge and the ability to apply functions.

Table 4-1. Latent variables linked to items, measured by particular tasks. The task numbers refer to Appendix A.

Latent variable name	Latent variable label	Item variable name	Item label	Tasks
ξ_1	Procedural knowledge of functions	x_1	Graphic procedures	1,2(2)
		x_2	Algebraic procedures	2(1),3,4,23
η_1	Conceptual knowledge of functions	y_1	Relations between graphic and algebraic representations	6,7,8,18
		y_2	Graphic interpretations	9,10,11,12
		y_3	Algebraic interpretations	14,15
η_2	The ability to apply functions	y_4	Economic applications	5,13(2),17,19
		y_5	Derivation	13(1),16
		y_6	Graphic knowledge of the derivative of a function	20,21

The intention of the pilot study was to detect the weaknesses or strengths of the different kinds of tasks. A complete analysis of the pilot study is not given separately, since the intention was to study the tasks as such, rather than the students' achievements. Instead, experiences from the pilot study are used to argue for the development of the tasks in the main test when appropriate. The tasks in the main test are discussed item per item, which means that all tasks that constitute an item are discussed in the same chapter. In Table 4-1, tasks are grouped per latent variable and item.

It is of course possible to design tasks, which require mainly procedural knowledge of functions. The difficulty lies in the development of tasks where the use of conceptual knowledge of functions is absent when the student works with them. If a student is asked to calculate the value of $x^2 + 3$ for $x = 2$ and he or she gets a negative answer, then he or she might conclude that the result is wrong because x^2 is non-negative and hence the answer cannot be less than three. In such a case the students might redo the calculation, based on judgments that include a type of knowledge, which can be considered as conceptual. However, since we do not look at types of knowledge as opposites, this does not cause severe problems for the analysis.

Tasks to measure conceptual knowledge of functions are designed in a manner, which makes it difficult to solve them by procedural knowledge alone. Those tasks are defined in such a way that the student is prohibited from a purely algorithmic solution strategy.

4.11 TASKS MEASURING PROCEDURAL KNOWLEDGE OF FUNCTIONS

Procedural knowledge of functions is probably reflecting the typical school mathematics focusing on skills. Kadijevich (1999) claims that these skills are primarily fostered through procedural tasks involving fully quantified objects. It is important to remember that this research is about knowledge of functions. Procedures on functions will obviously involve operations on concepts that are more elementary, such as integers.

Baker & Czarnocha (2002) investigated to what extent cognitive development is dependent on conceptual and procedural knowledge. They measured procedural knowledge through a student's average scores in mathematics. This confirms the impression that typical school mathematics, from which these scores originate, mostly reflects procedural knowledge. The questions in the test do not address practical applications, to avoid disturbance, and can be solved by following an algorithm. Presumably, no objectified or conceptual knowledge of functions is needed.

Tasks to measure procedural knowledge of functions are designed to test the students' ability to execute a plan, or more precisely to execute a procedure. As mentioned earlier, this relates to the third stage in Polya's list of four stages. Neither understanding the problem, nor conceiving a plan for its solution, should play a critical part. In other words, it should be trivial to understand what the student is asked for, and the problem of choosing between different solution strategies is kept to a minimum. Making choices between solution strategies can be seen as a process, which involves conceptual knowledge, since choices are often based on the judgment of properties or relationships.

The tasks can be solved using well-known algorithms and step-by-step procedures where the next step to be performed only depends on the state of the former and the solutions are possible to locate without seeking relation to other mathematical representations. If a problem is presented through a graph, no other mathematical representation of the problem is needed. Finally, tasks measuring procedural knowledge of functions should not be related to applications or subjects from other fields of research since applications are treated separately. Even if problems that involve such relations also require procedural skills, it would be hard to distinguish the types of knowledge the students applied to achieve their answers.

Two items measure procedural knowledge of functions, one related to the graphic procedures and the other to the algebraic procedures. The scores on the items are thought of as being caused by a common underlying factor, namely the procedural knowledge of functions, and the scores between the items are expected to correlate to some extent. On the other hand the two items will also account for the difference in graphic and algebraically skills, hence two different items.

4.11.1 Graphic procedures

This item (x_1 in Table 4-1), was designed to test whether the students are able to draw a graph, assuming that they had calculated values for a set of pairs (x, f(x)). An incorrect graph might originate from the fact that the student is unable to calculate the values of the function, or a lack of ability to draw the graph, even if the values for the pairs are correct. The following task was given in the pilot study:

Given the function $f(x) = 2x^2 - 8x + 6$, Df = R

Calculate the value of f(x) when x = -1 and when x = 4

Sketch the graph of f(x.)

Figure 4-9. Task from the pilot study measuring the ability to perform graphic procedures.

Only 2.2% of the students got the calculations in the first question wrong, but 17.6% failed to sketch the graph. It seems that the reason they got the graph wrong was caused by something else than errors in the calculations.

It is the second question, sketching the graph, which is tested in this item, while the first part belongs to item x_2 (algebraic procedures). This means that the scores are based on the procedure of sketching graphs. Similar questions were given in the main test, but with varying degrees of difficulty to account for the variation between students.

The questions to measure graphic procedures are:

Item	Task	Question
x_1	1	Sketch the graph of h(x) = 2x – 1
x_1	2(2)	The function g(x) is given by g(x) = $x + \frac{1}{x}$ Sketch the graph of g(x)

Figure 4-10. Tasks to measure graphic procedures.

Question 2(2) refers to the last part of task 2.[10] Even if the tasks are intended to address the portion of procedural knowledge that is related to graphs, the students have to calculate functional values algebraically before they sketch the graph. One could claim that the item contains an element of algebra, but since the calculations as such are considered trivial, the students are not expected to have many problems with this part of the task. Hence mistakes in calculations are not expected to be a dominant source of variation. The second task is expected to be more difficult than the first. Before answering the second question, students were asked to calculate different values for g(x), but these results are scored in item x_2. The fact that different values were calculated first makes it reasonable to believe that the remaining parts are purely procedural, i.e. plotting the points into a coordinate system and drawing the graph.

Hiebert and Lefevre (1986) characterize procedural steps as “production systems that require some sort of recognizable input for firing”. In this case the input is given as algebraic expressions that are familiar to the students. The objects that are operated upon are symbolic, and the students are assumed to have an objectified conception of them. Hence the input or “starting point” should be clearly understood. Symbols such as “x” and “+” are assumed to be familiar to the students. For a student to understand the statement of the problem (Polya’s stage 1), he or she must also have a clear idea of what the final state (a graph) is expected to be. That one has an idea of what a graph is, in general, can be taken for granted. However, being able to produce the particular graph is something else.

Conceiving a plan for the solution (Polya’s stage 2) should be unproblematic for the simple reason that the students have worked with similar problems a number of times. Of course, it is possible that some would try to produce the graph on a calculator, and then try to reproduce it by drawing the same shape on the paper. This is not a common approach, since students are familiar with the kind of answer that is expected.

[10] The first part, question 2(1), asking for calculations belongs to the item ‘algebraic procedures’.

In view of the previous comments, students who are unable to sketch the graph, may have failed in Polya's third stage, executing the plan. It might be that some have discovered points that were "out of the way" and did not fit smoothly with the other points, but got it right by calculating them again. This means that they were able to verify that parts of the answer were wrong (Polya's stage 4), but at least they have documented an ability to correct it. It is impossible to decide whether the student has done the reflection on the answers, causing the repetition of the procedure. However, we can say with an amount of certainty that those who got it wrong were not able to execute the algorithm.

4.11.2 Algebraic procedures

This item (x_2 in Table 4-1) probably represents the kind of procedures that the students have most experiences with from school, and many suggest a renewed focus on exploring patterns and seeking solutions rather than on just memorizing procedures and formulas (Schoenfeldt, 1982). For the purpose of measuring this item, the student's familiarity with such tasks makes it reasonable to believe that the first and second stage in Polya's model should not cause problems.

To explore how tasks could detect variation between students, the following questions were tested in the pilot study:

Given the functions:

$h(x) = e^{(x^2)}$

$k(x) = e^{3x}$

When is h(x) = k(x)?

Figure 4-11. Task from the pilot study to measure algebraic procedures.

Surprisingly, only 23.5% were able to solve this task correct. According to Hiebert and Lefevre (1986), both algorithms and symbol representation systems are parts of procedural knowledge. It is hard to find out whether the reason for failure is a lack of understanding of the formal language or symbols on one hand, or whether the students are unable to complete the algorithm on the other. Exponential functions and Euler's number were introduced to the students just a few weeks prior to the pilot test. The experience from this part of the pilot study was that recently introduced symbolism should be omitted to avoid difficulties in separating errors due to misunderstanding the symbolism and errors caused by lack of algorithmic skills.

Another weakness with the pilot study task, when it comes to measuring procedural knowledge of functions, was that the student had to realize that if the two expressions for the functions were equal then the exponents had to be the same. This requires recognizing conceptual properties related to exponential functions. So, performing this stage would require elements of conceptual knowledge. As a consequence, questions that presumed any kind of conceptual knowledge to select an appropriate solution strategy were omitted. This does not mean that there should be one and only one choice of algorithm for each question, but that it should be easy to find at least one algorithm that the student is familiar with to some degree.

The questions to measure algebraic procedures are shown in Figure 4-12.

Item	Task	Question
x_2	2	The function g(x) is given by g(x) = $x + \frac{1}{x}$ Calculate the value for g(x) when x = -5, x= -2 , x = -1, x = 1, x = 2 and x = 5
x_2	3	Given f(x) = -x – 3. For which value of x is f(x) = 0?
x_2	4	In this task we look at the function $f(x) = 2x^2 - 8x + 6$, D_f = R Calculate f(x) when x = -1 and when x = 4 When is f(x)=0? When is f(x)$\leq$0?
x_2	23	g(x) is a linear function. Write down the expression for g(x) when g(2)=0 and g(0)=4.

Figure 4-12. Tasks to measure algebraic procedures.

Even if the values can be calculated strictly by putting a value of x into the expression of the function, it is reasonable to believe that the students whose discourse is objectified, will detect an error due to wrong calculation more easily, by feeling that something is wrong with the answer and redo the calculation. The question remaining unanswered is not whether these students have better skills, but whether they have applied their conceptual knowledge when they were solving the problem. In tasks 2, 3 and 4 the procedures to follow are more or less given. In solving task 23, different procedural approaches are available. Since two points are given, one could insert the values for one point into g(x) = ax + b to find b and then use the other point to find a. Another approach is to insert values for both points into:

$$g(x)-g(x_1)=((g(x_2)-g(x_1))/(x_2-x_1)) \cdot (x-x_1)$$

to find g(x). One could argue that making a choice between different strategies is a mental activity in itself that requires more than just procedural knowledge. However, it is likely to believe that the students use the method they are used to, rather than making an in depth evaluation of which method is the most appropriate. Since both procedures require about the same amount of calculation, the probability of success is unlikely to be dependent on the choice of procedure. At least, both procedures have been taught in the classes at an earlier stage.

4.12 TASKS MEASURING CONCEPTUAL KNOWLEDGE OF FUNCTIONS

The items should cover the main aspects of the meaning of conceptual knowledge to meet the criteria for content validity (Bollen, 1989), in other words, they should measure what they intend to measure. We need to summarize the essence of the previous discussion on the nature of conceptual knowledge of functions as discussed in chapter 2.

The items should capture the aspect of reification (Sfard, 1991), that a concept can be understood as a unit, without thinking of the underlying procedures. Breidenbach et al. (1992) claim that the only way to make a mathematical object is to encapsulate a process.

The challenge is to create tasks in which the functions are treated as units and pose questions in such a way that a procedural approach does not work.

Another important aspect concerns *relations.* Hiebert and Lefevre (1986) describe conceptual knowledge as "a network in which the linking relationships are as prominent as the discrete pieces of information". This is in accord with "skilful drive along networks", appearing in the definition of conceptual knowledge on page 10. We can say that to possess in-depth conceptual knowledge is to be able to organize and structure content into a coherent whole by the judgment of a variety of relationships. Tasks where the challenge lies in linking different representation forms are included. There are other relational issues, as the ability to relate knowledge to other mathematical knowledge. This might be previous knowledge, such as knowledge of basic arithmetic, algebra and symbols, but also relations to more advanced concepts. As an example, tasks to relate functions to the derivative of functions are included.

Both the second and fourth stage in Polya's model involve judgement on relational issues. Conceiving a plan for the solution of a problem would be impossible without imagination or some idea of where the different strategies would lead. Planning takes place prior to the execution of the plan (stage 3) and will be based on something else than procedural steps.

The verification of the answer is also related to the properties of this concept. Assessing the answer means to evaluate whether the outcome is reasonable or not. The presence of control mechanisms is a characteristic for conceptual knowledge of functions. One will sometimes discover that a result of a procedure is unreasonable due to the fact that certain requirements or properties are not met. In such cases, the students are likely to try again. However, it is problematic to test students' capability to control the outcome of a task, except when they accept a false result that could have been detected.

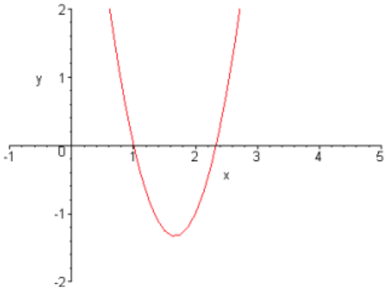
4.12.1 Relations between graphic and algebraic representations

This item (y_1 in Table 4-1) relates to different representation forms. The intention is not to test for algebraic skills and abilities to manipulate graphs separately, but to test whether students are able to see relations between the algebraic and graphic representations. Haapasalo (1993) distinguishes the levels of *concept identification* or *concept production*. He found that the latter types of tasks (i.e. tasks requiring production from one representation form to another one) are most reliable to measure conceptual knowledge. An essential question is whether students can make this transformation based as "skilful drive" or does they need to do it through procedural stages.

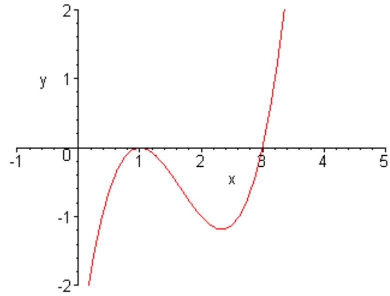
The pilot study included the task in Figure 4.13, being a typical identification task. The solution could be found by testing the zero points from the graphs with the algebraic expression for f(x). As the function represented by Graph 5 is the only one that is zero when x is 1, 2 or 3, this would lead to the correct answer. Since the students had worked with cubic functions prior to the test, this was expected to be a type of question that many students were able to master. The problem was that the answers, when registered as right or wrong, gave limited information. 83.8% of the students gave a correct answer, indicating that the majority saw these relations. The remaining 16.2% of the students either gave a wrong answer or did not answer at all. The small number of false answers was not enough to detect systematic patterns, and more questions were needed to account for variation between students.

Which graph belongs to the function $f(x) = x^3 - 6x^2 + 11x - 6$

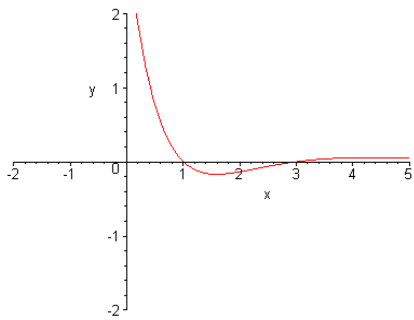
Graph 1



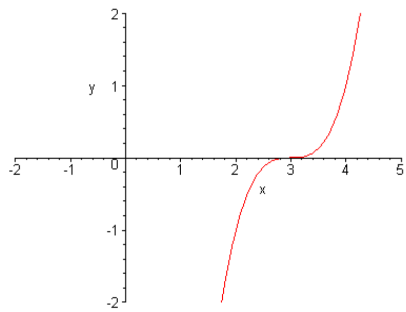
Graph 2



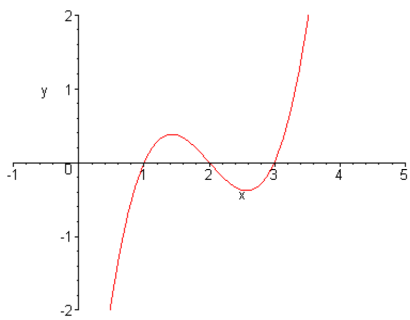
Graph 3



Graph 4



Graph 5



Graph 6

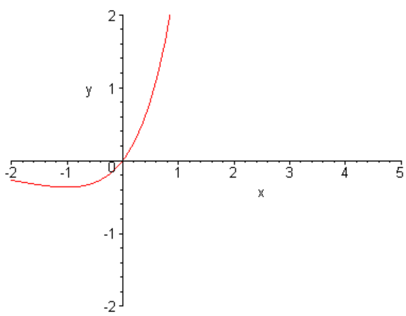


Figure 4-13. A task from the pilot study to identify algebraic and graphic expressions.

Another type of question from the pilot study was:

Below is a sketch of the graph of the function f(x). Write down the algebraic expression for f(x).

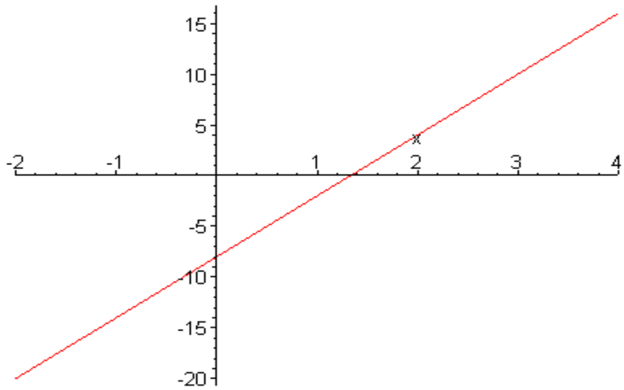


Figure 4-14. A production task from the pilot study to measure students' ability to produce an algebraic representation from a graphic one.

Only 61.8% managed to give a correct answer, which was lower than expected. The most common error was a wrong sign on the intercept constant. The task clearly addresses the link between graphic and algebraic representations, but, again, more questions were needed to give more detailed information.

At first sight, when we compare the two examples, one might expect a larger rate of success on the linear function problem. In general, cubic functions are at a more advanced mathematical level than linear functions, and the students are more familiar with linearity. Perhaps the explanation is that multiple-choice questions, as in the first example, challenge the students in a different way than in the second example. The first example mainly concerns the last stage in Polya's model. The calculations are already produced, and the students are left to verify the answers. In the second example, the students must conceive a plan for the solution, the second stage in that model. To quote David Tall (1991, p. 18) when he refers to the second stage in Polya's model: "The idea of 'devising a plan' is extremely daunting for the novice", which could explain the results.

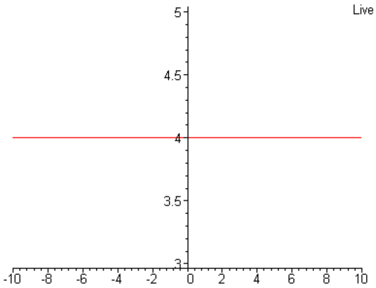
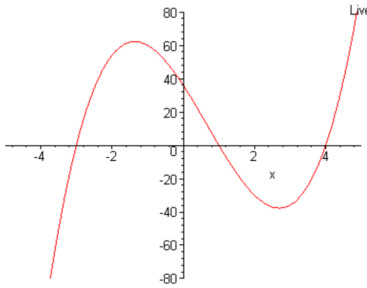
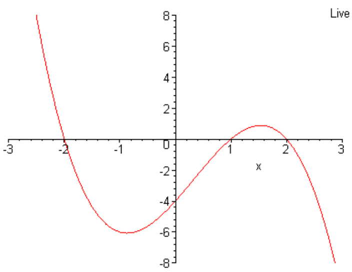
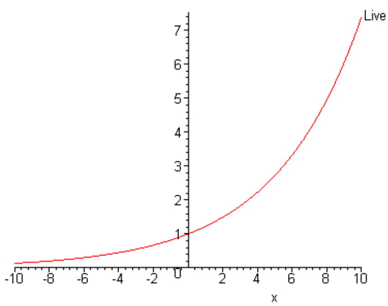
Item	Task	Question
y_1	6	The graph of f(x) is shown below. Write down the expression for f(x). 
y_1	7	The graph of a polynomial function is shown below  Which of the following expressions can the function be divided by? (x+1) (x+2) (x+3) (x+4) (x+5) (x-1) (x-2) (x-3) (x-4) (x-5)
y_1	8	A function of third degree has the form $f(x)=ax^3+bx^2+cx+d$. The graph of f(x) is sketched below. Find d. 
y_1	18	Use the graph below to decide when f(x)>2. You only need to write the answer 

Figure 4-15. A task to measure students' ability to work with different representations.

The experience from this part of the pilot was that a combination of questions involving verification and "traditional" questions seemed like a good idea, since it is important to cover the different aspects of conceptual knowledge of functions. The idea is to ask questions where the students reveal their ability to shift between different representation forms and thereby demonstrate knowledge of the isomorphism between the problem expressed in algebraic terms and the graphs. The questions to measure the transformation between graphic and algebraic representations are shown in Figure 4-15.

The solution of task 18 involves two steps. The first step is to draw a horizontal line through 2 at the vertical axis and then draw a vertical line through the intersection with the graph down on the horizontal axis. The lines can either be drawn physically, or just be thought of as a mental activity. Secondly, the answer must be expressed algebraically by reading the values from the x-axis and express the result correctly. An approach like the one described here can be regarded as procedural and easy to work through, once chosen. The most difficult part for the student is probably not to perform the procedural steps, but the process of choosing the right procedure, which requires some knowledge of the relationship between the different representation forms. Once again, Polya's model is a helpful tool to demonstrate how we can think about the nature of problems that may arise. It is likely that stage 2 is the most challenging phase to work through. The graph is used as an intermediate tool in the sense that the question is raised by use of a text including an algebraic expression. This is interpreted into the graph, where some procedures are performed. Thereafter the answer is expressed algebraically. The four tasks are different in content and also with respect to degree of difficulty, but do all address problems related to isomorphism.

4.12.2 Graphic interpretations

This item (y_2 in Table 4-1) concerns the encapsulation of a concept and is intended to determine how students are able to handle functions as units that could be operated upon, a characteristic for the objectification of a concept. It was necessary to raise problems where functions are represented as unified and compact as possible, but also in a way that does not enable a procedural solution strategy. The reason for representing the functions by graphs, as in these tasks, is that they contain enough information to solve the given problems, but lack information needed for a procedural approach. A graph is a type of carrier for the functional relationship (Dörfler, 1999), but does not provide the same details for procedural steps as an algebraic expression. Hence, graphic representations are probably the best way to represent functions as entities. The questions used to measure students' ability to work with these kinds of graphic interpretations are shown in Figure 4-16.

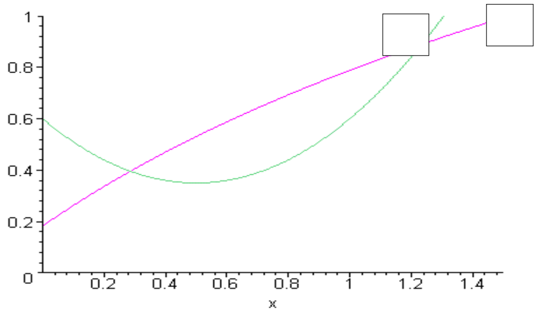
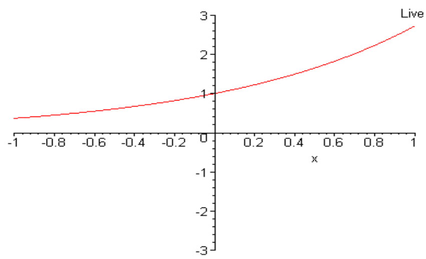
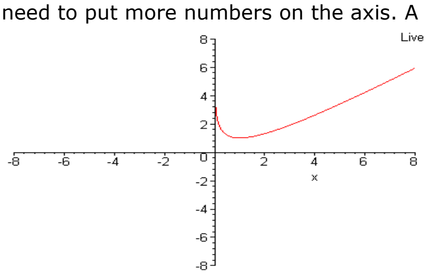
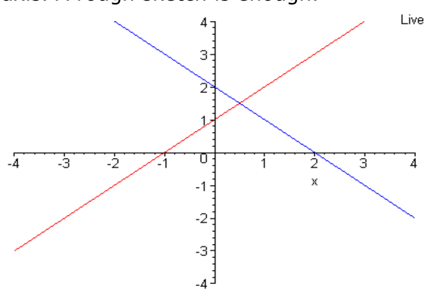
<table>
<tr><th>Item</th><th>Task</th><th>Question</th></tr>
<tr><td>y_2</td><td>9</td><td>Below you see the graphs of f(x) and g(x). Sketch the graph of the function f(x) – g(x). You don’t need to put more numbers on the axis. A rough sketch is enough.

</td></tr>
<tr><td>y_2</td><td>10</td><td>The graph of f(x) is shown below. Sketch the graph of f(-x). You don’t need to put more numbers on the axis. A rough sketch is enough.

</td></tr>
<tr><td>y_2</td><td>11</td><td>The graph of f(x) is shown below. Sketch the graph of -f(x). You don’t need to put more numbers on the axis. A rough sketch is enough.

</td></tr>
<tr><td>y_2</td><td>12</td><td>The graphs of two functions are shown below. Sketch the graph of the sum of the two functions. You don’t need to put more numbers on the axis. A rough sketch is enough.

</td></tr>
</table>

Figure 4-16. Tasks to measure graphic interpretations.

In tasks 9, 10 and 11, it is not easy, and very unlikely, that the students are able to determine an algebraic expression for the functions. Task 9 is quite similar to the examples than one often can find in economics where, as an example, functions for cost and income are drawn in the same coordinate system. The difference between them will be the profit. No problems were given regarding this similarity to economic subjects since the idea was to distinguish this item from items that measure the ability to apply functions.

The type of problem in tasks 10 and 11 are less familiar to the students. If the symbols –f(x) or f(-x) are not understood, the students will have problems to pass stage 1 in Polya's model. It is likely that some students have an idea of mirroring the graph around one of the axes or both. Pure guessing based on some intuition on mirroring would involve two solutions (x-axis and y-axis) or maybe mirroring around the origin.

Task 12 can be approached by several strategies, including manipulating directly the graphs. An alternative would be to find the algebraic expression for each linear function, add them, and draw the graph of the sum of the expressions. In both cases, the verification of the answer would most likely be done by inspecting the graphs.

4.12.3 Algebraic interpretations

In this item (y_3 in Table 4-1) functions are treated as units, being operated on as entities. No details are given that could lead to a procedural approach, as neither algebraic expression nor graphs are provided. The information that specifies characteristics about the functions is given in the text. The questions used to measure students' ability to work with algebraic interpretations on functions are:

Item	Task	Question
y_3	14	Suppose f(x) is a function of third degree and that g(x) is a linear function. What kind of function is h(x)=f(x)·g(x) ?
y_3	15	Suppose f(x) is a function of third degree and that g(x) is a function of second degree and that f(x) can be divided by g(x). What kind of function is j(x)=f(x)/g(x) ?

Figure 4-17. Tasks to measure algebraic interpretations.

The functions are not completely defined in the text, but in the same manner as item y_2, arithmetic operations where functions are entities that are operated on are addressed. The questions concern the degree of polynomial functions. Evidently students have knowledge of rules for potential expressions. Such tasks are linked to a part of the students' previous knowledgebase that has been established recently. The challenge is to conceive a plan for the solution by acknowledging that these rules must be applied.

Task 15 contains the information that f(x) can be divided by g(x). It is important to notice that the students had recently worked with polynomial division, which gives reason to believe that understanding the statement should not cause too much trouble.

Conceptual knowledge is often related to the ability to link pieces of information. If a student has a reasonable conception of fractions, rules for potential expressions and the symbolic representations of functions such as f(x), the questions above will require that the student is able to link the problems to his or her existing knowledgebase. Since the arithmetic rules most likely are well known to students, it is reasonable to assume that students who struggle with this task are unfamiliar with treating f(x), g(x) and h(x) as entities.

4.13 TASKS MEASURING THE ABILITY TO APPLY FUNCTIONS

In this study, the ability to apply functions as a concept is treated separately, not as an integrated part of conceptual knowledge of functions. This has also to do with a student's motives to learn mathematics. The motives are most likely that they need to use mathematics either in further studies of mathematics or in a field where mathematics is applied. Therefore the items that measure the ability to apply functions distinguish between application of functions "outside" and "inside" mathematics. It is important to emphasize that we do not study the ability of applying mathematics in general, as personal quality. Instead, we look at the capability of applying functions in particular. As an area of applications outside pure mathematics, it was natural to use examples from economics, while tasks in derivation measured the mathematical applications.

Once again, Polya's model is a suitable framework for comments. The first two phases, to understand the problem and plan the solution, must now be considered from many perspectives, of which functions is only one. It is about the ability to capture the complete picture of a situation, whose solution is determined by the use of functions in combination with elements from other areas. It seems reasonable to say that a student with a well-developed ability to apply mathematics will be able to apply a mathematical concept free of context, including contexts different from those in which the concept was taught. Some of the problems are presented in a way that differs slightly from what students are typically exposed to in the class.

It is difficult to design a test that is intended to measure the ability to apply functions without including questions that require calculations. The calculation part is skill-oriented, and does not address what we want to be included in these tasks. Therefore the test addresses problems that do not require advanced skills.

The theoretical rationale for using tasks in derivation for two of the items stems from the theory of Sfard (1991). She suggests that understanding develops from operational understanding to structural understanding, and further to structural understanding at a more advanced level. In this regard, one needs a structural understanding of functions to understand the derivative at an operational level, or similarly one needs a conceptual knowledge of functions to manage the derivative at a procedural level (Figure 4-18).

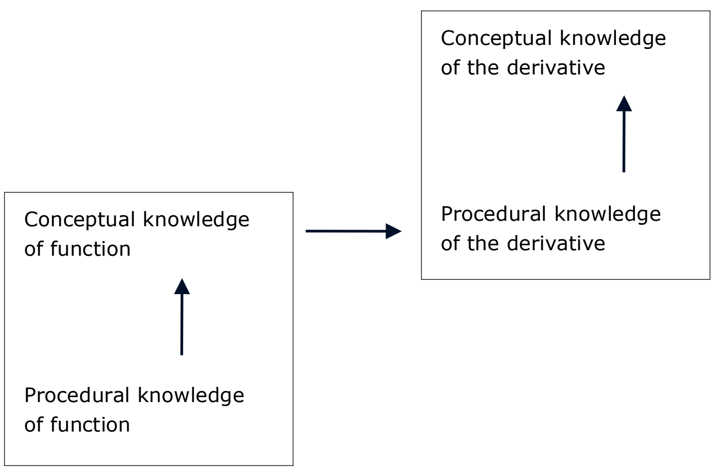


Figure 4-18. Development of knowledge via stages (indicated by arrows), following Sfard's theory.

Item y_4 relates to economic applications, focusing on problems related to cost and income. The two other items, y_5 and y_6, are both related to the derivative of a function, and are as such pure mathematical problems. This means that we measure the ability to apply the concept of a function on mathematical concept where we hypothesize that an objectified understanding of functions is needed. The questions solved algebraically and the questions requiring a graphic approach are split into two separate items, y_5 and y_6.

4.13.1 Economic applications

This item (y_4 in Table 4-1) addresses basic economic problems. The pilot study included the following problem:

> The demand for a product is a linear function of the product's price. The demand is reduced by 7 units when the price is increased by one NOK. When the price is set to 70 NOK, the demand is 210 units. Express the demand as a function of the price. What is the demand when the price is set to 77 NOK?

Figure 4-19. Problem from the pilot study addressing demand.

Only 47.8% of the students had a correct solution. A closer look revealed that very few set up an expression for the demand function. They had reached a correct answer, but the mathematical explanation was often incomplete. For example one student wrote 210-49 = 69. This suggests that students may be able to manage such problems without a symbolic representation like an algebraic expression or a graph. In fact, there was limited evidence on which steps the student followed to reach a solution. Even if teachers or researchers normally struggle to identify student approaches step by step, this is not critical in this context. What we look for is how the capability of reaching a correct solution is related to procedural and conceptual knowledge of functions. The conclusion of the experience from this part of the pilot-study was to include problems that are given textually and only require one answer. The questions used to measure economic applications are shown in Figure 4-20.

In task 5, the problem is raised textually and the student must find the solution by interpreting the text and regard salary as a function of the total costs. As commented on earlier, some students may be able to find the solution without expressing the solution in terms of functions.

The word 'function' is not used in the text, but what is important is the nature of the given problem. The text describes an initial state, that the cost for an employee is 500,000. In addition, the text gives some properties of the relation between an employee's salary and the company's total cost of employing a person. As such, typical properties of functions are embedded in the problem.

Task 13 asks the students to find out how the marginal cost can be calculated, given an algebraic expression for the cost. Given a cost function, they must make interpretations of it in an economic context. Two questions where the students are asked to give interpretations are also included. The intention is to assess the students' ability to connect the answer to an economic phenomenon.

In task 17, the problem can be solved without the graph. In fact it is very hard to read the result from the graph at all. The students are supposed to apply the expression for K(x) to deduce that the marginal cost is 90 and thereafter combine this result with the fact that the profit is at its maximum when the marginal income equals the marginal cost, which is for x=90. In this task, the student has a lot of information that is not required to solve the problem. A part of the challenge is to decide what information is relevant and what is not.

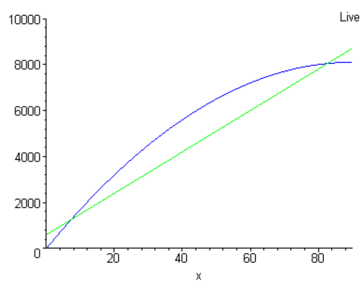
Item	Task	Question
y_4	5	The cost for a company to employ a person is the salary in addition to other costs (taxes) estimated to be 40% of the employee's salary. A company's total cost for an employee is 500.000. What is the salary?
y_4	13	The cost of producing x units of a product is given by $K(x) = -0{,}1x^2+6x+200$ when x is in the interval [0,20]. Estimate the marginal cost for x=10. What is the interpretation of this number? Estimate the marginal cost for x=15. What is the interpretation of this number in relation to the answer you got in the previous question?
y_4	17	The graphs of two functions are shown below. The linear function is a cost function giving the total cost by producing x units of a product and is given by K(x)=600 + 90x. The other graph shows the total income by selling x units of the product.  How much is the marginal income when the profit is at its maximum?
y_4	19	A company has a linear cost function. The cost of producing 15 units is 605 and the cost of producing 31 units is 877. What is the cost of producing 8 units?

Figure 4-20. Tasks to measure knowledge of economic applications of functions.

Finally, task 19 is in a similar category of questions as the one given in the pilot study. Hence the same comments account for this task as discussed previously. Again, the most important information is to see whether a correct answer is given. It might seem strange to disregard the strategy or procedure the students have applied, but that would involve a judgment of procedural capabilities that would disturb the analysis. Here, in this latent variable, the intention is just to see whether the answer is right, while distinguishing between the employment of procedural or conceptual knowledge of functions is left to the structural part of the model.

4.13.2 Derivation

In this item (y_5 in Table 4-1), students are asked to calculate the derivative of different functions. One might claim that this reveals only skills and not conceptual knowledge, but it is important to keep in mind that the operations are applied to produce a new function, namely the derivative, from a given function. Thus the functions themselves, as given in the tasks, are not the product of the operation, but rather the starting point. Similarly, one might claim that adding or multiplying rational numbers is also a skill oriented task, but according to the hypothesis proposed, integers need to be objectified to a certain extent to make possible operations on rational numbers.

The questions used to measure the item derivation are:

Item	Task	Question
y_5	13	The cost of producing x units of a product is given by $K(x) = -0{,}1x^2+6x+200$ when x is in the interval [0,20]. Calculate the marginal cost K'(x).
y_5	16	Calculate the derivative: a) f(x) = 4x + 2 b) g(x) = 3x4 + x2 −6x + 4 c) h(x) = $\frac{x^2 - 2x}{2x + 4}$ d) m(x) = ln(3x) e) n(x) = eax + b where a and b are constants

Figure 4-21. Tasks to measure derivation.

Task 13 could be interpreted as an economic application, but all the text regarding costs and units could in fact have been omitted. As such the two tasks are very similar, but this is not a theoretical problem, since they belong to the same item. What is important is to include tasks with various degree of difficulty. This is taken care of in task 16. Question *a* should be trivial to the students, while *d* and *e* are more demanding. Exponential and logarithmic functions were recently introduced to the students.

One can reflect upon whether the concept of a function is something that is understood in general, or whether the complexity of the function must be taken into consideration. In this regard, the complexity, or perhaps abstractness, of a function must be considered. It might be that students have much more trouble with recognizing the function in task 16 *d* as a function than they do with the one in task 16 *a*.

4.13.3 Graphic knowledge of the derivative of a function

Compared to the item y_5, this item (y_6 in Table 4-1) probably addresses a deeper knowledge of the derivative, since all representations are graphic. The core of the problem is to test relationships between a function and its derivative when both are drawn in the same coordinate system. Functions that were unknown to the students in the sense that they were not likely to find the algebraic expression for the function are included. As a consequence, it was practically impossible for them to find the algebraic expressions to check the answer in a procedural manner. It is likely that they approach such questions by trying out alternatives, in other words, with iterative use of stage four in Polya's model until the correct solution is found.

The following task was given in the pilot study:

Below you see two graphs in the same coordinate system. One of them belongs to the function f(x), and the other to f′(x). Decide which of A and B that belongs to f(x) and which belongs to f′(x).

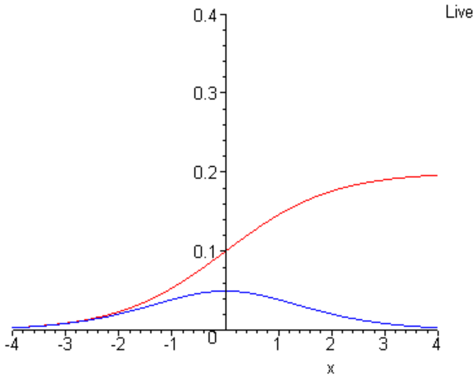


Figure 4-22. A task from the pilot study to distinguish the graphs of a function and its derivative.

Only 59.6% gave a correct answer. Keeping in mind that there are two alternatives, and that a pure guess would give approximately 50% correct answers, this told me that the students struggled with this kind of problem.

Another similar, but more complex task was:

Below you see three graphs in the same coordinate system. One belongs to the function f(x), the other to f′(x) and the third to f′′(x). Decide which one that belongs to f(x), f′(x) and f′′(x) respectively

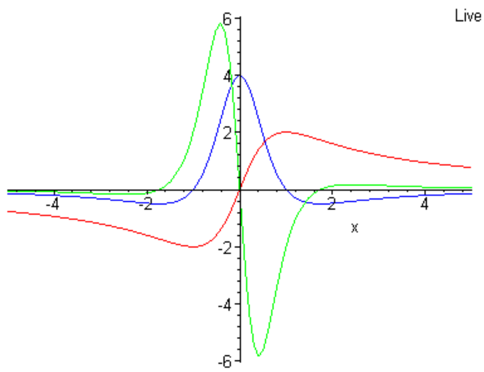


Figure 4-23. A task from the pilot study to distinguish the graphs of a function, its derivative, and its second order derivative.

23.5% managed to give the correct answer. This confirms the impression that the students' achievements are low in such tasks. Statistically, there are six possible answers to this task. Since it is difficult to interpret the results of those answers that are partially correct, questions with three graphs in the main test were omitted. Instead, several tasks comparable to the first example from the pilot study were included.

The questions used to measure graphic knowledge of the derivative of a function are shown in Figure 4-24.

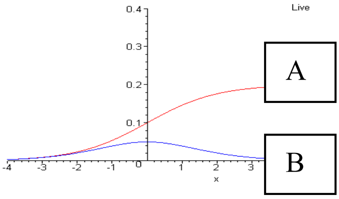
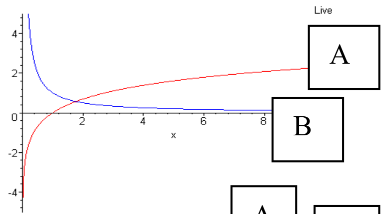
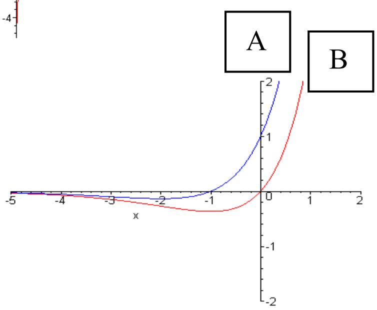
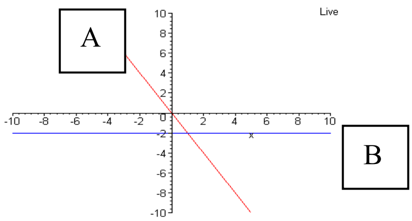
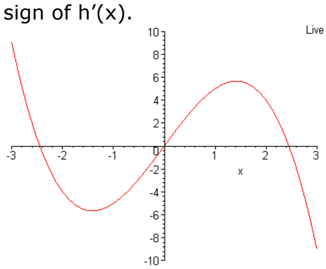
Item	Task	Question
y_6	20	The graph of a function f(x) and its derivative is shown in the same coordinate system. Decide whether A or B is the derivative. a)  b)  c)  d) 
y_6	21	The graph of the function h(x) is shown below. Fill in a schema for the sign of h'(x).  When is h'(x) largest?

Figure 4-24. Tasks to measure graphic knowledge of the derivation of a function.

Task 20 involves many possible reflections on the relationship between the function itself and its derivative. For example, in task 20 *a* one could deduce that B is the derivative from the fact that the derivative of A is at its largest value at x = 0. This requires the students to understand that the derivative is at its largest value when the slope of A is maximal. Thereafter, B must be recognized as a function of the slope of A, and that it has a maximum at x = 0. Even the fact that the slope and the value for the two graphs must be compared with respect to the same x-value is a kind of reflection that might not be obvious to all students. Several approaches are applicable to the questions in task 20. In question *c*, a possible path of argumentation could be that, since B has the value of zero at x = 0, it cannot be the derivative of A, because the slope of A is nonzero. Another approach would be to look at one of the graphs and try to find out what the derivative would look like, and see if it fits the other. To summarize, task 20 challenges the students to reflect on a variety of relationships between a function and its derivative. They must find the relevant information needed to be able to make judgments on the link between the two.

Task 21 involves two questions. The first is intended to test whether the students understand the relationship between the slope of the graph and the sign of its derivative. The other question can be answered by investigating the graph. It might reveal whether students confuse the maximum of the function with the maximum of its derivative.

4.14 POST TEST

One of the weaknesses of validity discussions and fit estimates[11], not only in structural equations models, but also in other analyses as for example linear regression, is that the measures are often evaluated by applying the same data that are used to estimate the model parameters. One way avoiding this problem in this study was to collect a new set of data. The aim of the post test is to compare the outcome of the statistical analysis to the outcome from another group of students, but with a different statistical approach. The intention of the post test is to evaluate the validity of the measures in the main test by comparing the test results with exam performances and to see whether the data from the post test confirm the relationships between the different types of knowledge found in the structural part of the model analysis. The aim of the post test is not to provide new evidence, but to see if the sample from the post test would strengthen or weaken the findings from the main test.

The post test was conducted on 57 first year students in Trondheim taking the same course in mathematics as the students in the main test, but the structural equation model was not run on the post-test data since the number of students was too low for a meaningful analysis. Data from the post test was collected by giving the students the same tasks as in the main test and the scores on each task were registered according to the same procedure as in the main test. In addition, the students were identified by their student code. For each student, three index scores were estimated (equations (4.16) and (4.17) based on the factor loadings from the model that was estimated in chapter 5.

[11] R2 in ordinary least squares linear regression is normally estimated by means of the same set of data used to estimate the regression parameters.

$$z_1 = \sum_i \lambda_i x_i \qquad (4.16)$$

$$z_j = \sum_i \lambda_i y_i \text{ for } j = 2,3 \qquad (4.17)$$

Index scores on procedural knowledge of functions, conceptual knowledge of functions and the ability to apply functions were used to investigate whether these measures could predict students' performances at their exams in related topics. At the end of the same semester, three types of performances in related topics were investigated from the exams. The results from two exams, one in mathematics and one in business economics were used for this purpose. The exam in mathematics was a multiple choice exam covering basic algebra and elementary function theory, while the exam in business economics was a regular written exam. The student identification number was used as a key to compare each student's performance on the post test with his or her performance at the exams. Figure 4-24 illustrates how the index scores from the test were held up against the performances from the exams.

Procedural and conceptual performances in mathematics were registered by achievements from specific tasks from the multiple-choice exam in mathematics, while application performance was indicated by the grades from the exam in business economics. If the measures of the latent variables are valid, then the index scores should predict the corresponding outcomes from the exams to a certain extent.

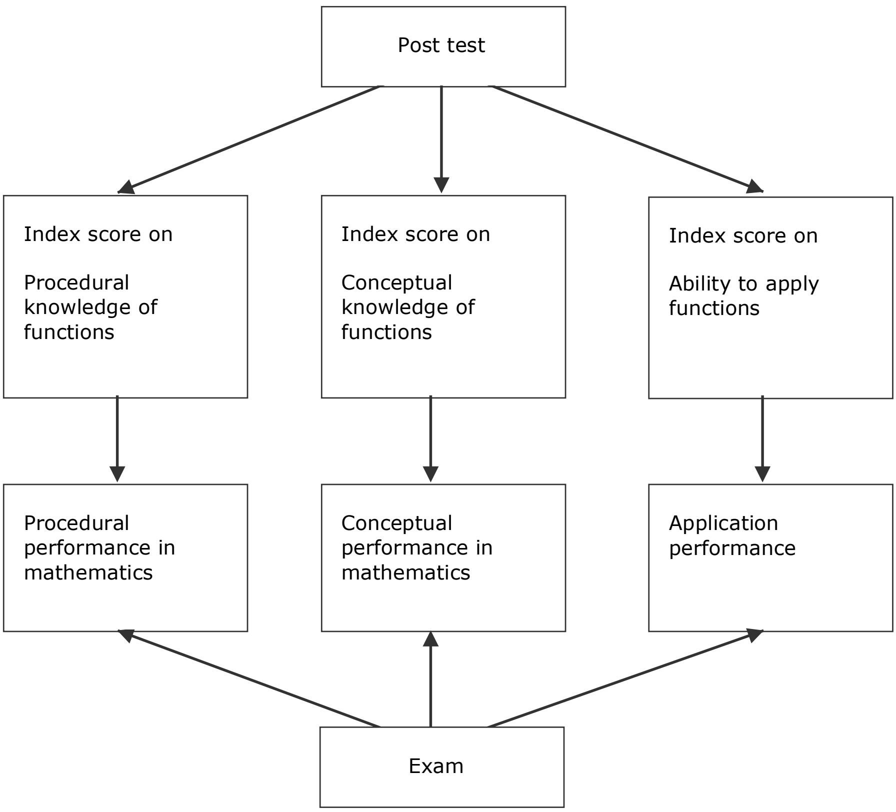


Figure 4-25. Relationship between index scores from the test and exam performances.

The index scores were also used as variables in regression analysis to study the conclusions on research questions 2 and 3 related to dependencies between the three different types of knowledge.

4.15 INTERVIEW

Of course there are factors not included in the statistical analysis that might have an impact on conceptual development and achievements. Use of technology, students' approaches to learning or teachers' teaching strategies are not subject to the statistical analysis. The statistical analysis does not reveal anything about the pedagogical approach used when the subjects studied the conceptual and procedural knowledge involved in the main test. Neither does it reveal anything about how mathematical beliefs and educational background among the subjects influenced their outcomes. Understanding that beliefs as subjective knowledge affect a person's actions and behaviour (Maass & Schlöglmann, 2009), it is likely that students' beliefs are shaped not only by their school background but also by the beliefs and actions from their teachers' side. Thus, interviews were used to find out some possible explanations for the outcomes.

Because of technical reasons, direct interviews among the subjects and their teachers were not possible. Therefore, three students from different upper secondary schools in Norway were chosen to form a quasi peer group. They did not participate in doing the whole main test but took the very same course in mathematics as those students who were the subjects of the main test. In addition to a semi-structured interview, stimulated recall was used to find out how they solved the tasks of the main test and how they described their solution process. Two from those three students, called here Emma and Anna, had had mathematics for two years at the upper secondary school, whilst the third one, called Martin, had had three years advanced syllabus with mathematics.

5 Results

This chapter presents descriptive statistics from the main test, such as mean and standard deviation on the data at aggregated level for each item. Thereafter an estimation of model parameters and a modification of the model are shown. Finally, a result from the post test and experiences from the interviews are outlined. It is important to be aware that an inferential analysis in the structural equation model, as used in this study, emphasizes covariation rather than ordinary measures of location and scatter. In other words, the primary focus is directed on students at an aggregated level rather than as individuals, but some comments on students' responses are included.

5.1 CALCULATION OF SCORES

Each task was scored on a range, which is shown in Table 5-1.

Table 5-1. Scoring range of the tasks.

Task	Scale	Item
1	0-4	x_1
2(1)	0-6	x_2
2(2)	0-5	x_1
3	0-4	x_2
4	0-12	x_2
5	0-4	y_4
6	0-4	y_1
7	0-6	y_1
8	0-4	y_1
9	0-5	y_2
10	0-5	y_2
11	0-5	y_2
12	0-4	y_2
13(1)	0-3	y_5
13(2)	0-9	y_4
14	0-4	y_3
15	0-4	y_3
16	0-20	y_5
17	0-4	y_4
18	0-4	y_1
19	0-8	y_4
20	0-16	y_6
21	0-8	y_6
23	0-4	x_2

It is important to keep in mind that the differences in the ranges do not affect the primary analysis in the study. Even if item x_1 only gives 9 points if all answers are correct and x_2 can give as much as 26 points, this does not mean that x_1 has smaller impact than x_2. The point is that we are concerned with variation, not the level of the scores. Therefore the results are independent of the range of the scale. However, in the descriptive analysis below, the total score will be more influenced by the items with high maximum scores. A score was also calculated for each item by adding the scores of the tasks that belong to the item. Table 5-2 shows the scale for each item.

Table 5-2. Scale for scores aggregated on each item.

Item	x_1	x_2	y_1	y_2	y_3	y_4	y_5	y_6
Scale	0-9	0-26	0-18	0-19	0-8	0-25	0-23	0-24

5.2 TOTAL SCORE

A total "sum" was estimated for each student as the sum of scores from each task. Figure 5-1 shows that the distribution[12] of the total score among the students is fairly close to a normal distribution with a mean score equal to 75.6. None of the students managed to get the maximum possible score that is 152. The highest score among the students is 142, while the lowest is 9. It might seem obvious that the total score follows a normal distribution, but students' different levels of mathematics from their school background, might have disturbed the symmetry. On the basis of earlier experience, the impression is that students with specialization[13] in mathematics at high school achieve significantly better results in mathematics than the rest of the students. It seems that this has not caused skewness as far as the total score is concerned.

The mean total score on all students is 75.6, which is 49.7% of the possible maximum. The standard deviation is 30.8 points, which corresponds to 20% of the possible maximum. For the purpose of the statistical analysis in this study, these numbers are satisfying since the intention is to account for variation among students. The difference in achievements would be very hard to detect in a test where the average score is very low or very high. The fact that the variation also is quite large indicates that the set of tasks includes questions with different degrees of difficulty.

[12] The frequency refers to the number of students that achieved the different scores on "sum".

[13] Specialization in mathematics means that the student learned mathematics for three year in high school.

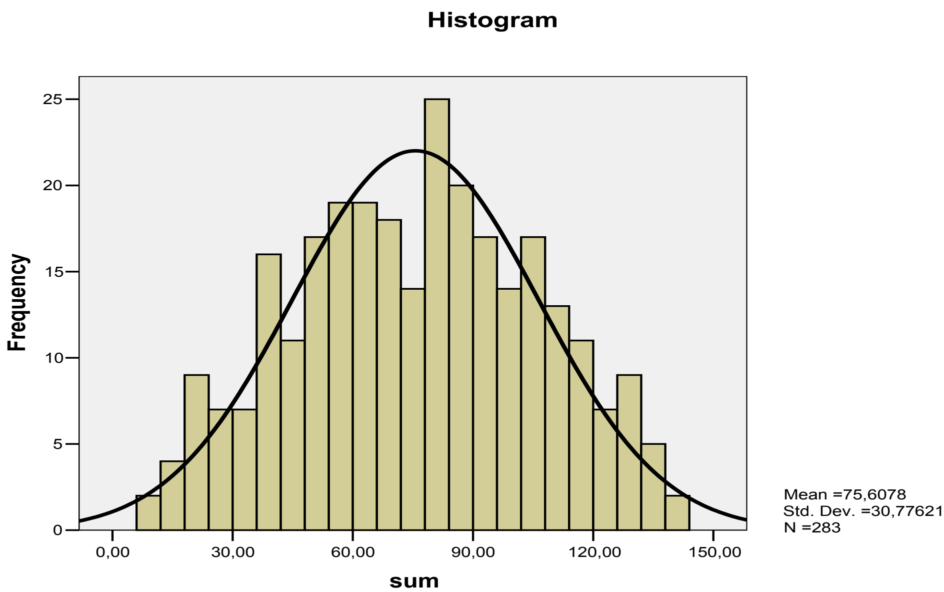


Figure 5-1. Distribution of the sum variable of the test.

One should be careful to conclude on differences between part-time and full-time students based on this material, since the number of students in the part-time classes is relatively low, as shown in Table 5-3.

Table 5-3. Mean score per group of students including standard deviation to indicate dispersion.

	Mean	Std Deviation	Count
Sandefjord part-time	46,77	32,92	13
Sandefjord full-time	75,22	31,39	27
Lillestrøm part-time	54,71	25,95	21
Oslo B full-time	77,10	29,51	105
Oslo C full-time	81,32	29,42	117
Group Total	75,61	30,78	283

The bar chart in Figure 5-2 shows that the part-time students had lower scores than full - time students, but the variation is larger among the part-time students as indicated by the 95% confidence interval for the errors.

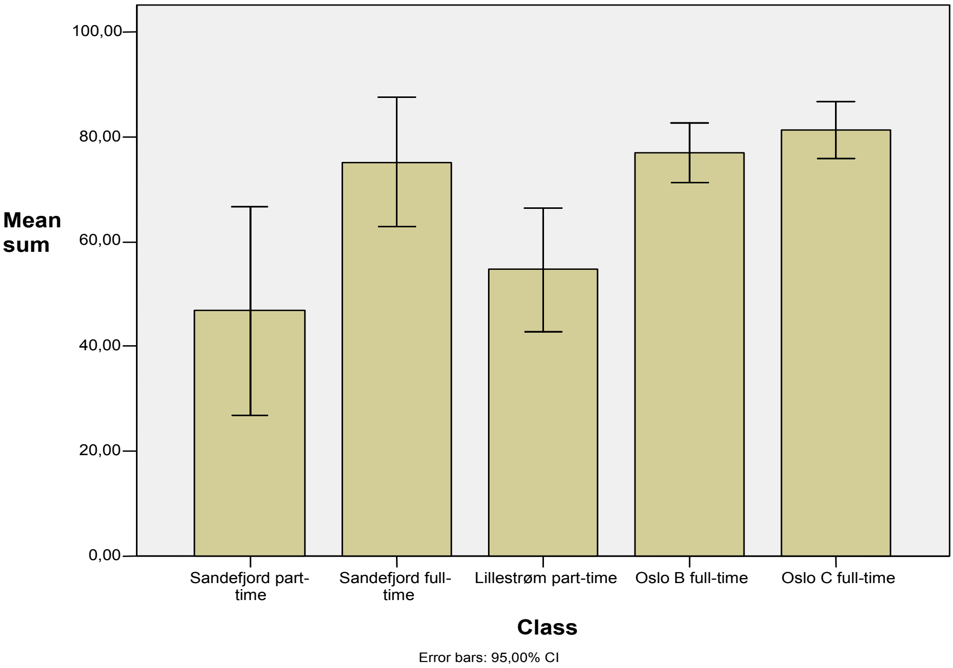


Figure 5-2. Mean scores of each group including confidence interval (95%) for the mean.

It is possible that the larger variation in score among the part-time students may be traced back to the large variation in the school and professional background in these groups.

5.3 SCORES PER ITEM

A table of descriptive statistics for mean and standard deviation is provided for each item. The tables contain statistics for each task belonging to the item, as well as statistics for the item. The results can be commented from two perspectives, either from the student perspective or from the task perspective. From a student perspective, the obvious interpretation is to see whether they scored well or badly as a group, and whether there were large differences in performance between the students. From a task perspective, the results can indicate whether the task is suitable for this type of analysis. Here the standard deviation plays a more important role than the mean since the results in structural equation model origins from variation and covariation, rather than location. The comments in this chapter address both perspectives.

5.3.1 Graphic procedures

Overall the students scored 6.31 out of 9 points (70.1%) on average, which seems quite satisfying, but the questions are relatively simple and straightforward (Table 5-4).

Table 5-4. Results for tasks measuring item x_1.

	N	Minimum	Maximum	Mean	Std. Deviation
Task 1	283	0	4	3,37	1,304
Task 2	283	0	5	2,95	2,337
Item x1	283	0	9	6,31	2,882
Valid N	283				

In the first task students were asked to sketch a linear function. This is not expected to cause much problem. The second task is about the graph of a rational function, and the variation in answers between the students is larger than in the first task. In the second task, a typical error is to draw one curve by joining the two segments like in Figure 5-3.

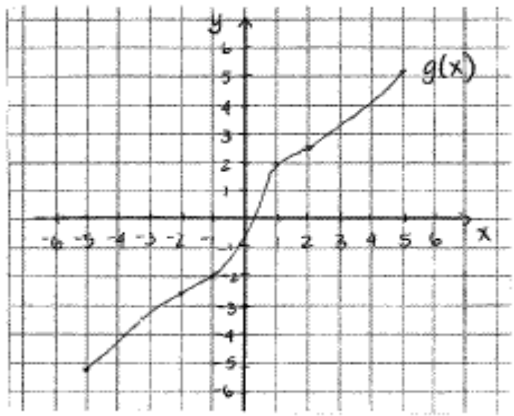


Figure 5-3. Typical student error discarding the two different branches of the function.

One could argue that this kind of error is caused by lack of conceptual knowledge. A student who is aware that the function is undefined for x=0, and therefore the graph cannot intersect the y-axis, should detect such errors. However regardless of this, it is reasonable to believe that a student with well developed skills on drawing graphs is more likely to succeed in this task than a less skilled student. It is maybe a bit surprising that the difference in achievements between tasks 1 and 2 did not differ more since drawing a hyperbola is far more complicated than drawing a straight line.

5.3.2 Algebraic procedures

Also for this item, the mean total score is quite high, 18.66 of 26 (71.8%). The first three tasks gave far better average scores than task 23, which also had the largest standard deviation when we adjust for the range of the scale (Table 5-5). In the first tasks, the algorithms involved consist of just a few steps. In task 23, where the question is to write down the expression for a linear function where two points are given, involves some more steps. No obvious systematic errors seem to occur among the erroneous solutions.

Table 5-5. Results for tasks measuring item x_2.

	N	Minimum	Maximum	Mean	Std. Deviation
Task 2	283	0	6	5	1,702
Task 3	283	0	4	3,45	1,302
Task 4	283	0	12	8,60	3,828
Task 23	283	0	4	1,26	1,808
Item X2	283	0	26	18,66	6,370
Valid N	283				

5.3.3 Relations between graphic and algebraic representations

Table 8-6 shows that apart from task 6, the mean score is less than 50% of the maximum score for the other tasks. The reason for this can be that these types of questions are somewhat different from the more traditional tasks the students are familiar with. In addition the relatively low scores are due to the fact that conceptual knowledge represents a more advanced type of knowledge.

Table 5-6. Results for tasks measuring item y_1.

	N	Minimum	Maximum	Mean	Std. Deviation
Task 6	283	0	4	3,37	1,427
Task 7	283	0	6	1,94	2,472
Task 8	283	0	4	1,62	1,952
Task 18	283	0	4	1,85	1,907
Item Y1	283	0	18	8,78	5,377
Valid N	283				

In task 7 the students were asked to decide which of the given linear expressions the cubic function is divisible with. The function was only represented by its graph. A closer look at the answers detected that a typical error is related to sign. The function intersected the x-axis at x=1, and is therefore divisible by x-1, while many students answered that it is divisible by x+1. It seems that the students have a tendency to rely on their intuition when they struggle to understand what is really going on. The other tasks did not reveal systematic patterns in wrong answers.

Overall the item contains a balance between easy and more difficult questions as can be seen from the results of tasks 4 and 6. In addition to the magnitude of the standard deviations, this is an indication that the questions reveal variation among students' achievements.

5.3.4 Graphic interpretations

The last task (12) was considered slightly easier than the others, and it had the highest scores (Table 5-7). The most surprising results came from tasks 10 and 11, where the students were asked to sketch the graph of f(-x) and -f(x), respectively, given the graph of f(x). It was expected that task 10 would cause more problems than task 11, as the symbol f(-x) is maybe more unfamiliar to the students than -f(x). Despite this a priori assumption, the scores on task 11 were considerably lower. The experience was in many ways the same as in the pilot study that the students have some ideas of mirroring the graphs, but often fail to get it correct.

Another comment on the results for this item is that the values for standard deviation, adjusted for the range of the scale, were relatively high. If we focus on the task perspective, the results support the suitability of such tasks in analysis where one looks for variance between students' achievements. Even if variation is detected, it is obvious that students seem to have problems with the graphic interpretations. The total mean of 4.09 of 19 possible points (21.5%) leaves little doubt of this.

Table 5-7. Results for tasks measuring the item y_2.

	N	Minimum	Maximum	Mean	Std. Deviation
Task 9	283	0	5	1,13	2,069
Task 10	283	0	5	1,12	2,077
Task 11	283	0	5	,57	1,586
Task 12	283	0	4	1,26	1,801
Item Y2	283	0	19	4,09	4,645
Valid N	283				

5.3.5 Algebraic interpretations

Table 5-8 represents statistics of the tasks related to the degree of polynomial functions, 53.1% of the student answers being correct. The students who gave detailed explanation, without exception, concluded with correct answers. The erroneous answers were mostly single word answers. As expected, there was a tendency that the answers were correlated (r=0,734). Some judgment had to be done, when the score was assigned to the answer. One example is that instead of saying that a function is a polynomial function of degree four in task 14, some students just said that it is a polynomial function. It is not a question of remembering the right words. The idea is that meaning, such as meaning connected to the degree of a function, is embedded in the words that name them.

Table 5-8. Results for tasks measuring item y_3.

	N	Minimum	Maximum	Mean	Std. Deviation
Task 14	283	0	4	2,17	1,958
Task 15	283	0	4	2,08	1,931
Item Y3	283	0	8	4,25	3,621
Valid N	283				

5.3.6 Economic applications

The challenge in solving the problem in task 5, where the salary is a function of the total cost related to an employee, can be divided into three parts. The problem is given as a text, and most of the students tried to set up a mathematical model in terms of an equation that describes the situation. The next challenge is to solve the equation, while the third is to reflect on the answer. The following table represents the statistics.

Table 5-9. Results for tasks measuring item y_4.

	N	Minimum	Maximum	Mean	Std. Deviation
Task 5	28	0	4	1,52	1,941
Task 13	28	0	9	5,28	3,446
Task 17	28	0	4	,64	1,424
Task 19	28	0	8	2,42	3,586
Item Y4	28	0	25	9,86	6,975
Valid N	28				

A look at the results revealed a systematic misunderstanding that can be associated with the first and last of the steps in the problem solving process. Figure 5-4 shows a typical example where the calculation is right, but the algorithm is wrong.

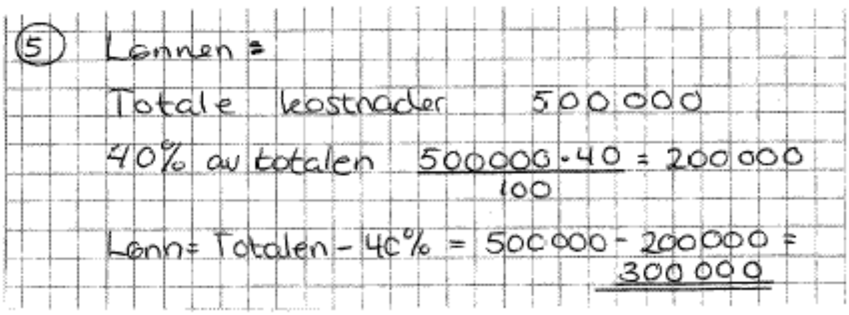


Figure 5-4. An example of correct calculations with a wrong algorithm.

Thus, the equation has been defined but the equation does not reflect the problem correctly. The model is specified wrong, and that a critical reflection on the result is either missing or has led to wrong conclusions. The situation can be described by Figure 5-5. It seems that mistakes can be illustrated by arrows crossing the horizontal line and separating the problem of economics from the problem of mathematics. In other words, it seems that the problems connected to economic applications are related to the specification of a model, which in mathematical terms reflects the economic problem and also the ability to reflect on the result. Referring to Polya's model, the challenges relate to stages 1, 2 and 4.

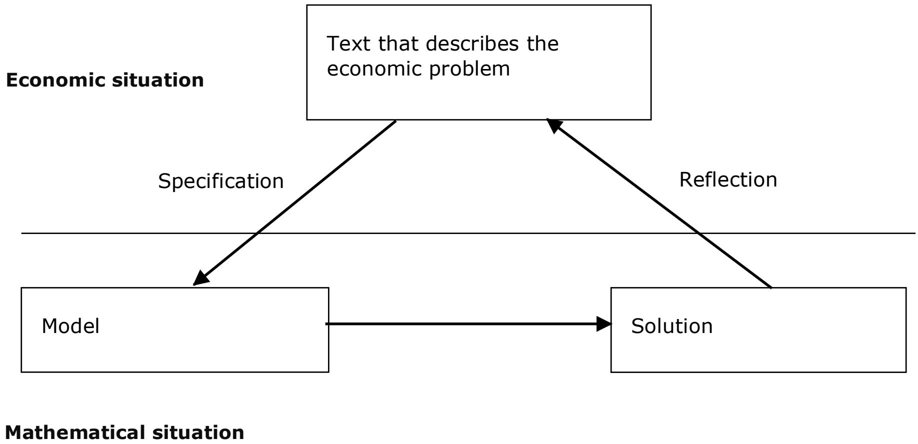


Figure 5-5. Solution of an economic problem through mathematical modelling.

In task 13, many students had problems with the interpretation of the results, as the mean score indicates. This has also to do with reflection, as reflection is not only associated with the ability to detect errors, but with the students capabilities to give a textual interpretation of the estimated answers.

Task 17 also gave lower score than expected. The problem involves several economic terms; total cost, marginal cost, marginal income and profit. Most of the student who had a correct answer had realized the fact that the profit reaches its maximum when the marginal cost (K'(x)) equals the marginal income and therefore that the answer is 90.

The answers to task 19 showed a lot of different strategies that reminded of trial and error strategies. The information is provided textually, and a majority of the students are struggling to specify a mathematical model by use of graphs or algebraic expressions. It seems that the problem is to approach and solve the problem with mathematical tools, in other words to apply functions. However, some students used this approach successfully. In this regard such tasks are suitable for the purpose of this kind of analysis.

5.3.7 Derivation

Questions *a* and *b* from task 16 were both correct answered by more than 90% of the students, while question *d* caused problems for many. One might argue that these tasks reflect skills rather than understanding, but a closer look at the results from the different parts of task 16 shows that the students had most difficulties with questions *d* and *e* which involves few procedural steps (see tables below). Maybe it is possible that one of the reasons that students have problem with derivation of logarithmic and exponential functions, despite the relatively simple algorithms required, is that they have not grasped the idea of these types of functions.

Table 5-10. Results for tasks measuring the item y_5.

	N	Minimum	Maximum	Mean	Std. Deviation
Task 13	283	0	3	2,37	1,179
Task 16	283	0	20	12,80	4,821
item Y5	283	0	23	15,18	5,305
Valid N	283				

Table 5-11. Specification of the results for task 16.

	N	Minimum	Maximum	Mean	Std. Deviation
Task 16 a	283	0	4	3,82	,822
Task 16 b	283	0	4	3,67	1,050
Task 16 c	283	0	4	2,24	1,626
Task 16 d	283	0	4	1,37	1,872
Task 16 e	283	0	4	1,70	1,976
Valid N	283				

5.3.8 Graphic knowledge of the derivative of a function

Two questions related to task 20 are: What is an appropriate solution strategy and how can one verify the suggested solution? It is impossible to decide whether errors can be assigned to the lack of an appropriate solution strategy, or insufficient knowledge of the properties of the derivative. Yet it may be reasonable to believe that the latter is the dominating cause of error.

As seen in Table 5-12, task 21 caused a lot of problems, especially the second part of question 21, which is only correctly answered by 10% of the students. Three of Polya's stages are present in the tasks measuring item y_6. Question 20 addresses, as already discussed, the second and the fourth stage. In task 21 one might suspect that a great number of the students have halted already at the first stage and understood the problem.

Despite the low mean score on item y_6, the relatively high standard deviation indicates that the tasks distinguish between students' achievements.

Table 5-12. Results for tasks measuring the item y_6.

	N	Minimum	Maximum	Mean	Std. Deviation
Task 20	283	0	16	7,10	5,783
Task 21	283	0	8	1,38	2,489
Item Y6	283	0	24	8,47	7,271
Valid N)	283				

5.4 MODEL PARADIGM

The analysis of the model is performed with two objectives. One objective refers to the structural part of the initially hypothesised model, hereafter referred to as the original model, and is related to research question 2 concerning how different types of knowledge are related to each other. The second objective is a discussion on the quality of the measurement model. Does the operationalization of the measures reflect the meaning we want them to represent, and is the accuracy of the measurement instrument sufficient? These questions are addressed by analysing results from the main test as well as from the post test. The analysis of the original and competing models were estimated in LISREL by use of Maximum Likelihood estimators and the covariance matrix. The results and comments are based on this analysis, but the main statistics for the analysis when other estimators were used is also included. Results are also shown when the covariance matrix is replaced with the correlation matrix. The initially hypothesised model is based on the assumption that procedural knowledge is a necessary condition for conceptual knowledge of function. A separate analysis is performed on the model to decide if this causal direction is supported, or whether the data indicate that the model should be adjusted by a reverse of bidirectional link as indicated in Figure 3-5.

Structural equation modelling, as applied here, follows the paradigm of hypothesis testing, but a short explanation of how it is implemented is given first. A more detailed overview of the theoretical background for the model is given in Appendix B.

The observations from the main test are basically a matrix of correlations or covariances between item scores for all pairs of items. For simplicity it is referred to covariance matrix in the following, even if the same applies for correlation matrixes. The notation, often referred to as the LISREL notation, from Jöreskog (K. G. Jöreskog, 1973, 1977), is used. The idea is to create a model, represented by a set of parameters, θ, that predict the observed covariance matrix, S, in the best possible way. We have the following definitions:

Σ	is the population covariance matrix
θ	is a vector that contains the parameters in the model
$\Sigma(\theta)$	is the covariance matrix as a function of θ[14]
$\hat{\theta}$	is the vector that contains the estimation of the parameters
$\Sigma(\hat{\theta})$	is the estimated covariance matrix
S	is the observed covariance matrix

[14] A correct t model with the correct parameters, would reproduce the covariance matrix exactly (Bollen, 1989).

Since the idea is to identify a model that reproduces the population covariance matrix in a best possible manner, our hypothesis is therefore:

H_0: $\sum(\theta)=\sum$
H_1: $\sum(\theta)\neq\sum$

A fit-function F evaluates how well the observed covariances (S) fit the estimated covariance matrix ($\sum(\hat{\theta})$), and a test statistic is derived from this function as described later in this chapter. If this statistic is small, H_0 is likely to be true and the model fits the data well, while a large value supports the alternative hypothesis H_1. This is a slightly different orientation than what usually is the case with the testing of hypothesis, as keeping H_0 gives support for our model. The idea is, in other words, to create a model that predicts the data, here being the inter-item variance/covariance matrix as good as possible.

Item scores on all 283 students from the main test were put in a spreadsheet and processed in PRELIS (K. G. Jöreskog & Sörbom, 1988), a module in LISREL for processing raw data. The most important output from PRELIS is the variance/covariance, or alternatively the correlation matrix, between the items. The correlation matrix has the advantage that the factor loadings can be compared with respect to their size, since they are scale independent. This means that the highest loading has more impact on the factor (latent variables) than the lower. On the other hand, if interpretations of the factor loadings are desirable, the covariance matrix estimates are preferred (Hair, Anderson, Tatham, & Black, 1998, p. 603). In that case the estimated parameters must be interpreted with respect to the items' scale of measurement.

Table 5-13 and Table 5-14 present the covariance and correlation matrixes, respectively.

Table 5-13. The covariance matrix. The diagonal displays the variance of the scores for each item, while each of the other elements gives the covariance between two different items.

	x_1	x_2	y_1	y_2	y_3	y_4	y_5	y_6
x_1	8.31							
x_2	8.25	40.58						
y_1	5.97	20.85	28.91					
y_2	3.35	11.51	11.30	21.58				
y_3	3.27	12.34	10.17	7.46	13.11			
y_4	6.50	25.15	21.35	14.38	12.47	48.65		
y_5	4.52	17.39	12.58	7.35	8.10	19.89	28.15	
y_6	5.87	20.83	17.71	13.84	11.77	23.26	15.10	52.87

Table 5-14. The correlation matrix.

	x_1	x_2	y_1	y_2	y_3	y_4	y_5	y_6
x_1	1							
x_2	0.449	1						
y_1	0.385	0.609	1					
y_2	0.250	0.389	0.452	1				
y_3	0.314	0.535	0.523	0.444	1			
y_4	0.323	0.566	0.569	0.444	0.494	1		
y_5	0.295	0.514	0.441	0.298	0.422	0.537	1	
y_6	0.280	0.450	0.453	0.410	0.447	0.459	0.391	1

The table shows that each item is perfectly correlated with itself. Hence, all diagonal elements are one. Each of the other elements gives the Pearson correlation between the scores on two different items. Not surprisingly, all estimated correlations are positive, which is expected since the high performers among the students are expected to score better on all types of questions. The question is whether the entire model, with possible adjustments, will be able to reproduce a correlation matrix that fits closely to the observed matrix above.

There are several fit measures available in LISREL and it is necessary to distinguish between the theoretical and the empirical fit. The theoretical fit can be described as the degree of isomorphism between a theoretical model and a true model, while the empirical fit evaluates how well the observed covariances (S) fits the model generated covariance matrix $\Sigma(\hat{\theta})$. The goodness of the model is discussed primarily by using one of the most accepted empirical fit measures applying a test based on the Chi-square statistics. A large value for Chi-square indicates a bad fit whereas a small value for Chi-square indicates that the model fits the data well. The other goodness of fit statistics, which is used for the analysis, is also discussed in this chapter. If F denotes a fit function, the fit indexes (Chi-square, RMSEA, NFI) are derived from Min $F(S, \Sigma(\hat{\theta}))$ (Olsson, Troye, & Howell, 1999). Brown & Cudeck (1993) claim that rather than to ask whether the fit is correct, it is sensible to assess the degree of the lack of fit. If the sample size is large, it is likely that the hypothesis will be rejected, even if the model approximates the covariance matrix reasonably well. Still, our goal is to have a well-specified model that fits the data as well as possible. The minimum of the fit function will depend not only on how well the model is specified, but also on the estimation method.

5.5 MODEL IDENTIFICATION

This chapter will discuss the different estimators to be applied in this study and their choice.

A model needs to have a positive number of degrees of freedom to make possible the generalization of the results. This is called an over-identified model. In this study we have hypothesised an over-identified model with 17 degrees of freedom for the original model and 18 degrees of freedom when adjustments were made to the model. An over-identified model with positive degrees of freedom is required so that we are able to estimate the parameters. LISREL performs a test to check for under-identification and did not report any problems concerning identification, neither for the original nor the adjusted model. The conclusion is that the input matrix provides enough information to estimate all the parameters requested in the model.

LISREL offers several methods for the estimation of parameters. Maximum likelihood (ML), generalized least squares (GLS), unweighted least squares (ULS) and weighted least squares (WLS) are some of the most frequently used estimators[15]. The assumptions for the different estimators differ, and it is important to identify the most appropriate method for the data that was collected in the test. The performance, in terms of fit, is affected by sample size, specification error and kurtosis (Olsson et al., 1999). The estimation based on maximum likelihood estimators was used in this analysis, but the parameter estimates when other estimators are applied is also presented.

Why was ML preferred in this case? The ML estimators are asymptotically unbiased and therefore hold in large samples. Olsson, Troye & Howell (1999) conclude that ML tends in general not only to be more stable, but it also demonstrates higher accuracy both in terms of empirical and theoretical fit compared to GLS and WLS. GLS allows small samples, but it requires that the model should be well specified. A mis-specified model might well give misleading conclusions. An important characteristic of the ML, GLS and WLS estimators is that they are invariant and free with respect to scale. In this study the different items have different scales in the sense that they have different ranges in scores. As an example x_1 is scored on a scale from 0 to 9, while x_2 is score from 0 to 26 (Table 5-1). Being invariant means that it is invariant to change measurement units to one or more of the variables, while freeness means that the estimators are invariant to linear transformations of the scales. This allows the use of correlation matrix instead of the covariance matrix. WLS requires large data, and has the property that it seems to give better fit when the kurtosis is high, i.e. the more peaked the data are. However, the goodness of fit seems to be at the cost of inaccurate parameter estimates. The ULS fit function is relatively easy to understand, but a disadvantage is that it is neither invariant nor free with respect to scale. Altogether ML seems to be the most appropriate estimation method for this study, estimates will also be calculated with GLS, ULS and WLS. If the estimates are close when different estimators are applied, it will count as support for the model and the estimates.

The fit functions for the different methods that are applied in this study are (Bollen, 1989, pp. 334,425):

Maximum likelihood fit function:

$$F_{ML} = \log|\mathbf{\Sigma(\theta)}| + tr\left\{\mathbf{S\Sigma(\theta)^{-1}}\right\} - \log|\mathbf{S}| - (p+q) \qquad (5.1)$$

where p is the number of endogenous variables and q is the number of exogenous variables.

Generalized least squares fit function:

$$F_{GLS} = \left(\frac{1}{2}\right) tr\left(\left\{\mathbf{I} - \mathbf{\Sigma(\theta)S^{-1}}\right\}^2\right) \qquad (5.2)$$

Unweighted least squares fit function:

$$F_{ULS} = \left(\frac{1}{2}\right) tr\left[\left(\mathbf{S} - \mathbf{\Sigma(\theta)}\right)^2\right] \qquad (5.3)$$

[15] In statistical terminology a method for estimation of parameters is called an estimator.

Weighted least squares fit function[16]

$$F_{WLS} = \left[\mathbf{s} - \boldsymbol{\sigma}(\boldsymbol{\theta})\right]' \mathbf{W}^{-1} \left[\mathbf{s} - \boldsymbol{\sigma}(\boldsymbol{\theta})\right] \quad (5.4)$$

Fit measures can be estimated independent of the selected estimator.

5.5.1 The initially hypothesised model

Figure 5-6 gives the output path diagram with the estimated parameters and the most commonly used fit statistics when maximum likelihood estimators were applied. The three latent variables are named "proc" (ξ_1) for procedural knowledge of functions, "conc" (η_1) for conceptual knowledge of functions and "apply" (η_2) for the ability to apply functions.

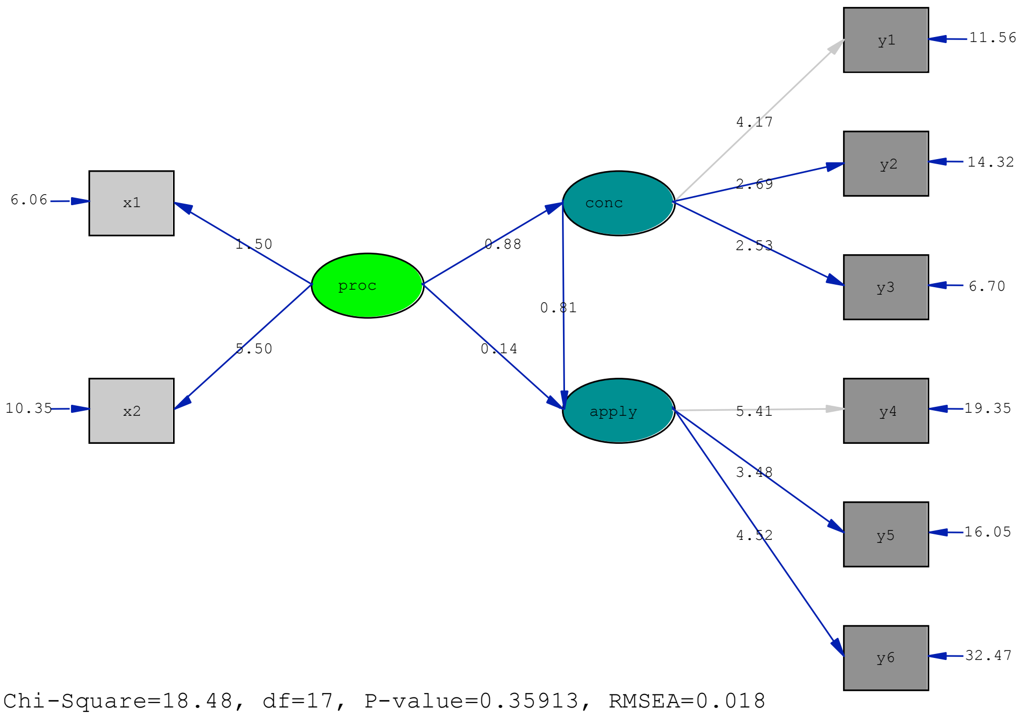


Figure 5-6. The LISREL output of the complete model estimated with Maximum Likelihood estimator.

In Figure 5-6, altogether 19 estimates of parameters are given. As seen from the path diagrams, the parameters are estimated, and can be investigated one by one in addition to examining the entire model. A first look at the diagram clearly indicates that the model fits the data reasonably well with a p-value of approximately 0.36, which is well above

[16] s is a vector of (1/2)(p+q)(p+q+1) elements by placing the non-duplicated elements of S in a vector, and σ(θ) is the corresponding vector of Σ(θ). W is (1/2)(p+q)(p+q+1)x(1/2)(p+q)(p+q+1) positive definite weight matrix (Bollen, 1989).

0.05. If the model holds, then the model-generated matrix will be equal to the true matrix. The hypothesis tested by the Chi square test is:

H_0: $\sum(\theta)=\sum$
H_1: $\sum(\theta)\neq\sum$

In our tests the resulting p-value is 0.36 and H_0 cannot be rejected. As H_0 supports the model, this result is in favour of our a priori assumptions. A closer look at the structural regression equations in Figure 5-7 indicates that most of the relationships are as expected, but reveals a weakness with the model in the sense that one relationship is questionable, being the relationship between procedural knowledge of functions and the ability to apply functions.

```
                 Structural Equations

  conc = 0.88*proc, Errorvar.= 0.22  , R² = 0.78
        (0.080)              (0.095)
         11.04                2.33

  apply = 0.81*conc + 0.14*proc, Errorvar.= 0.12  , R² = 0.88
        (0.24)       (0.22)              (0.067)
         3.43         0.62                1.78
```

Figure 5-7. The latent variable model displaying the dependencies between procedural knowledge of functions (proc), conceptual knowledge of functions (conc) and the ability to apply functions (apply) as estimated regression equations in the competing model.

The correspondence between procedural knowledge of functions and conceptual knowledge of functions as well as the relation between conceptual knowledge of functions and the ability to apply functions seems strong. This is in favour of the assumptions underlying the model, even if it does not serve as a proof for causality. In the first equation in the latent variable model, procedural knowledge of functions is significant at any reasonable level with t = 11.04. The same is true for conceptual knowledge of functions as an exploratory variable for the ability to apply function, as can be seen in the second equation. The total effect of procedural knowledge of function on the ability to apply functions can be decomposed:

$$\text{Total effect} = \text{direct effect} + \text{indirect effect} = 0.14 + 0.81*0.88 = 0.14 + 0.71$$

The t-value for procedural knowledge of functions, as an independent explanatory variable in the second regression equation, is not significant with t = 0.62. This could suggest that procedural knowledge of function is redundant, and could be left out as an explanatory variable for the ability to apply functions. In that case we would have a competing model were the arrow suggesting a direct effect between procedural knowledge of functions and the ability to apply functions is left out. This result indicated that an adjustment to the model should be considered. It is a requirement that an adjustment of the model should not result in a significantly weaker model fit. A test to compare models where one model is a special case of another was applied (Fornell & Larcker, 1981). For such a test to be meaningful the models must be nested and the selected model should have a structure that enables us to study the research questions. An analysis of a comparison between the competing model and the original model is required.

5.5.2 The competing model

The adjusted model, hereafter named the competing model, still contains the same factors measuring the three different types of knowledge, but the direct link between procedural knowledge of functions and the ability to apply functions is removed.

Figures 5-8 and 5-9 show the latent variable structure in the two models. The models are said to be nested meaning that the competing model with fewer estimated relationships is nested within the original model. This means that the direct relationship between procedural knowledge of functions and the ability to apply functions is fixed to zero in the competing model.

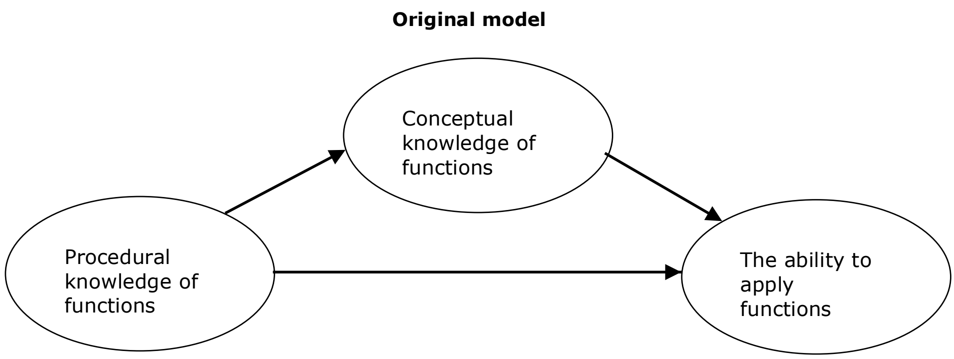


Figure 5-8. Latent variable model as primarily hypothesised.

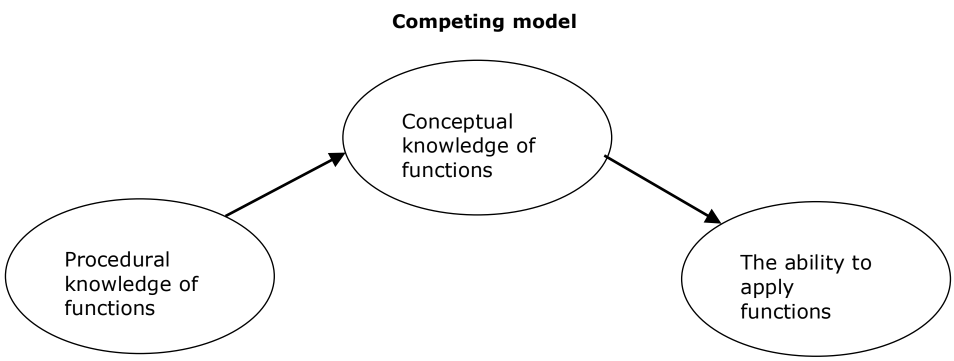


Figure 5-9. Latent variable model for the competing model.

An assumption for comparing models by a Chi-square test is that the models being compared are nested. Nested models have the same number of items and latent variables, but only differ in the number of parameters to be estimated (Hair et al., 1998, p. 591). In this case, these conditions are satisfied. In this study the originally hypothesised model is said to be less parsimonious than the competing model. A chi square estimate for the less parsimonious model is nearly always lower than for the more parsimonious model (Bollen, 1989, p. 270). Here, the difference between the models, being the difference in chi square, is tested for significance with one degree of freedom being the difference in the number of parameters to be estimated (Hair et al., 1998). However, it is important that the preferred model has a structure that complies with the research questions. Research questions 1 and 2 are unaffected by the change suggested in the competing model, but research question 3 will be approached in a different manner. The third research question is:

How does the ability to apply functions relate to procedural and conceptual knowledge of functions?

If the direct link between procedural and conceptual knowledge of functions is weak, or even absent, the role of conceptual knowledge seems even more critical.

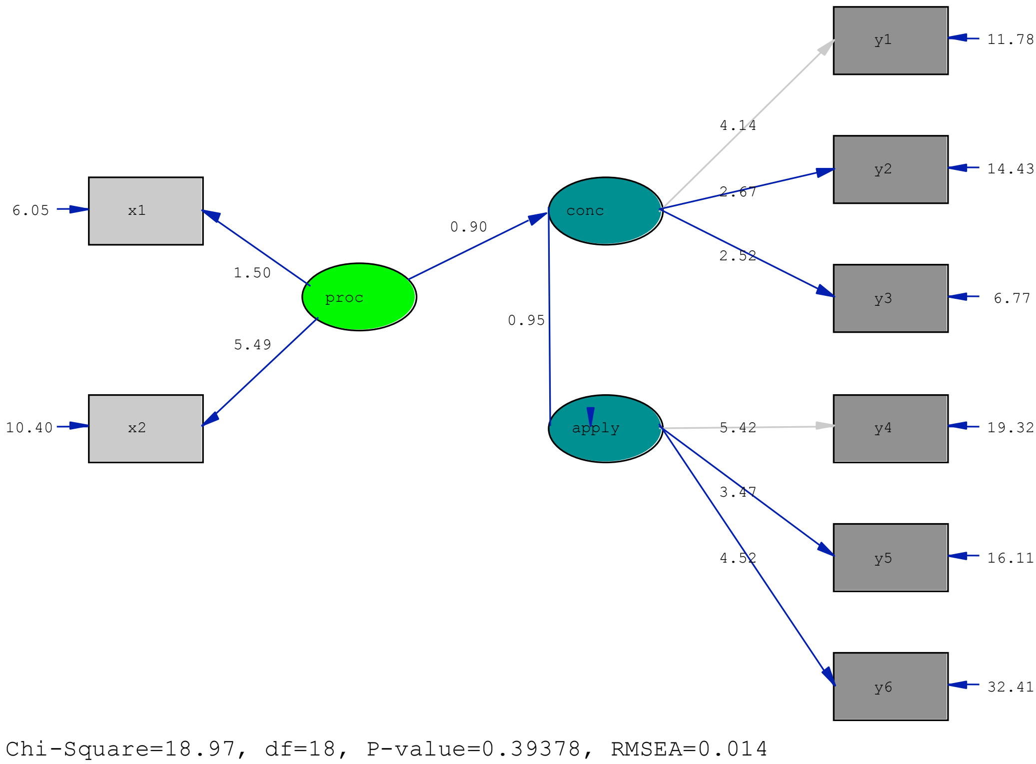


Figure 5-10. Estimation from LISREL of the competing model.

Figure 5-10 shows the path diagram for the competing model. Procedural knowledge of functions does not seem to contribute to the ability to apply functions in other ways than through conceptual knowledge. The path diagram with the estimated parameters shows the value of RMSEA equal to 0.014 and a p-value of approximately 0.39. In the same way as in the original model, this model shows a significant relation between procedural and conceptual knowledge as well as one between conceptual knowledge and the ability to apply functions. The difference in Chi-square between the two models is low (Table 5-15).

Table 5-15. Chi-square estimates of the original and the competing model.

	Chi-square	degrees of freedom
Competing model	18.97	18
Original model	18.48	17
Change	0.49	1

According to Fornell and Larker (1981), the less parsimonious model (the original model) should only be supported if the Chi square is significantly better when adjusted for the change in degrees of freedom. The estimated Chi-square for the competing model is 18.97 with 18 degrees of freedom while the Chi-square is 18.48 with 17 degrees of freedom for the original model. The change in Chi square (0.49) is lower than the critical value for a chi square test with one degree of freedom (3.84 at 5% level), and consequently the competing model is preferred. Small differences in chi square measures do not provide enough information to allow choosing between models, but the more parsimonious model should in general be supported. According to Bagozzi and Yi (1988), small changes in chi square are merely due to capitalization of chance and the restrictions are supported. The output from LISREL also provides other statistics for the comparison of models that support the competing model[17].

One should be careful in changing the hypothesized model since the assumption on which we base our competing model is no longer made a priori, but is rather based on results from the estimation of the first model. Thus one might argue that our competing model is based on a conditional assumption, violating a basic principle in inferential statistics. On the other hand, one cannot be sure that the first model is the best, and some room for alternatives should be allowed. The items and the measurement models are identical in the two models. The only difference is that one relation is left out in the competing model, meaning that the only difference is related to the latent variables.

5.6 EVALUATION OF THE MODEL

Sometimes the estimated model might provide offending estimates in terms of negative error variances or standardized coefficients larger than one. The first requirement is to investigate the estimates in this respect. Figure 5-11 gives the estimates related to the measurement model. As seen from this part of the LISREL output, no negative error variances are reported. For each item, except for items y_1 and y_4, where the parameters were fixed, a standard error (in parenthesis) and t-value is reported. Problems with very large standard errors, which would give large t-values, do not seem to occur. In fact the smallest t-value is 8.62, which is significant at any reasonable level.

The output for the structural model in Figure 9-6 below does not reveal any problems with negative error variance or large standard errors.

[17] ECVI, AIC and CAIC belong to this category (K. Jöreskog & Sörbom, 1993).The smallest values indicates the best fit. For the original model the values are respectively 0.20, 56.48 and 144.74. For the competing model the values are 0.19, 54.97 and 138.59

```
LISREL Estimates (Maximum Likelihood)
Structural Equations

  conc = 0.90*proc, Errorvar.= 0.20  , Rý = 0.80
        (0.080)              (0.085)
         11.22                2.34

 apply = 0.95*conc, Errorvar.= 0.10  , Rý = 0.90
        (0.079)               (0.062)
         11.93                 1.63
```

Figure 5-11. The latent variable model displaying the dependencies between procedural knowledge of functions (proc), conceptual knowledge of functions (conc) and the ability to apply functions (apply) as estimated regression equations in the competing model.

The estimates for the regression parameters have increased slightly when compared with the estimates for the original model. Procedural knowledge of functions is still a clearly significant explanatory variable for conceptual knowledge of functions with t = 11.22. Also, conceptual knowledge of functions obviously explains the ability to apply functions with t = 11.93.

```
LISREL Estimates (Maximum Likelihood)
Measurement Equations

     y1 = 4.14*conc, Errorvar.= 11.78, R² = 0.59
                             (1.29)
                              9.10
     y2 = 2.67*conc, Errorvar.= 14.43, R² = 0.33
        (0.29)               (1.32)
         9.30                10.96
     y3 = 2.52*conc, Errorvar.= 6.77 , R² = 0.48
        (0.22)               (0.67)
         11.38                10.12
     y4 = 5.42*apply, Errorvar.= 19.32, R² = 0.60
                             (2.37)
                              8.14
     y5 = 3.47*apply, Errorvar.= 16.11, R² = 0.43
        (0.33)              (1.58)
         10.48               10.20
     y6 = 4.52*apply, Errorvar.= 32.41, R² = 0.39
        (0.45)              (3.09)
         9.94                10.48
     x1 = 1.50*proc, Errorvar.= 6.05 , R² = 0.27
        (0.17)              (0.55)
         8.62                10.98
     x2 = 5.49*proc, Errorvar.= 10.40, R² = 0.74
        (0.39)              (2.95)
         13.91               3.53
```

Figure 5-12. The measurement model displaying how each item is explained by the factors (proc), (conc) and (apply).

5.7 NORMALITY

Structural equation models are to some extent sensitive to the distribution characteristics of the variables with respect to normality. The models assume absence of strong skewness and kurtosis[18] (Hair et al., 1998, p. 106). In addition to univariate normality for each item, the model also assumes multivariate normality (Bollen, 1989, p. 418). Statistics for skewness and kurtosis, to test for univariate normality, are provided in Table 5-16. Curran et al. (1996) considers the item scores to be moderately non-normal if the absolute value of skewness is in the range from 2.0 to 3.0 and similarly if the kurtosis is in the range from 7.0 to 21.0.

Table 5-16. Test of univariate normality for continuous variables.

Variable	Skewness	Skewness z-score	Skewness p-value	Kurtosis	Kurtosis z-score	Kurtosis p-value
X1	-0.572	-3.752	0.000	-0.911	-5.913	0.000
X2	-1.073	-6.286	0.000	0.682	1.995	0.046
Y1	0.184	1.280	0.201	-1.002	-7.299	0.000
Y2	1.117	6.478	0.000	0.746	2.127	0.033
Y3	-0.113	-0.786	0.432	-1.800	40.306	0.000
Y4	0.282	1.943	0.052	-0.814	-4.763	0.000
Y5	-0.554	-3.648	0.000	0.319	1.122	0.262
Y6	0.299	2.052	0.040	-1.019	-7.594	0.000

Several variables have significant skewness and kurtosis and consequently the tests for multivariate normality are rejected as shown in Table 5-17. It might be that the estimation is influenced by non-normality. However, violation of normality does not affect the consistency of the maximum likelihood estimates of θ, but excessive kurtosis can cause inaccuracy in the Chi-square items.

Table 5-17. Test of multivariate normality for continuous variables.

Skewness			Kurtosis			Skewness and Kurtosis	
Value	z-score	p-value	Value	z-score	p-value	Chi Square	p-value
5.853	7.483	0.000	77.764	-1.133	0.257	57.271	0.000

When non-normality threatens the maximum likelihood estimators, one possible correction is to employ an alternative estimator. Weighted Least Squares (WLS) is one such estimator that allows for non-normality (Bollen, 1989, p. 245). The weighted least squares estimates gave Chi-square = 20.60 with p-value=0.30 (Table 5-18). Keeping in mind that the p-value should exceed 0.05, we can conclude that the model still holds.

[18] With a normal distribution skewness and kurtosis should be equal to zero. Negative skewness indicates skewness towards the right and vice versa. A peaked distribution gives positive kurtosis.

5.8 ALTERNATIVE ESTIMATION METHODS

Even if Maximum Likelihood is the preferred estimation method, estimates of the parameters by means of some of the other most common estimators are also provided. As the main results in Table 5-18 show, the variation in estimates and fit measures is relatively small. None of the estimates would change any of the primary conclusions. All models have RMSEA < 0.05, non-significant p-values ($p > 0.05$) for the Chi-square tests. The factor loadings and regression coefficients also seem non-sensitive to the estimation method. Analysis is also performed on the correlation matrix and it indicates that the main results are independent of the matrix type. Overall, the estimates did not seem to vary much between the estimators or the type of input matrix. All estimations indicate that the model fits the data reasonably well. As mentioned earlier, these results are satisfying as they support the robustness of the model.

Table 5-18. The estimated values using Maximum Likelihood (ML), Unweighted Least Squares (ULS), Weighted Least Squares (WLS) and Generalized Least Squares (GLS) applying covariances and correlations.

	Covariance Matrix			**Correlation Matrix**		
	ML	**ULS**	**WLS[19]**	**ML**	**ULS**	**GLS[20]**
Chi-Square	18.97	18.90	20.60	18.97	18.90	21.00
p-value	0.39	0.40	0.30	0.39	0.40	0.28
RMSEA	0.014	0.013	0.023	0.014	0.013	0.024
γ_{11}	0.90	0.89	0.91	0.90	0.89	0.91
β_{21}	0.95	0.97	0.95	0.95	0.95	0.94
λ_1	1.50	1.50	1.51	0.52	0.52	0.52
λ_2	5.49	5.51	5.31	0.86	0.86	0.86
λ_3	4.14	4.11	4.11	0.77	0.77	0.77
λ_4	2.67	2.70	2.57	0.58	0.58	0.58
λ_5	2.52	2.53	2.63	0.70	0.70	0.69
λ_6	5.42	5.35	5.50	0.78	0.77	0.78
λ_7	3.47	3.47	3.43	0.65	0.64	0.66
λ_8	4.52	4.48	4.56	0.62	0.63	0.62

5.9 THE MEASUREMENT MODEL

Even if the fit measures give an acceptable overall fit, the measurement part needs to be investigated separately. Whereas the measurement model is related to fit measures and measures for reliability, the magnitude of the regression parameters in the latent variable model is important for the research questions 2 and 3.

What we try to do in the measurement model is to measure concepts such as procedural knowledge of functions. It is important to have a measure that corresponds to the meaning associated with a concept and that the measurement of the latent variables is sound. Reliability is the consistency of the measurement (Bollen, 1989, p. 206), while validity expresses whether we measure what we intend to measure.

[19] WLS utilizes the asymptotic covariance matrix and was not run on correlations.
[20] GLS was only run on correlation matrix.

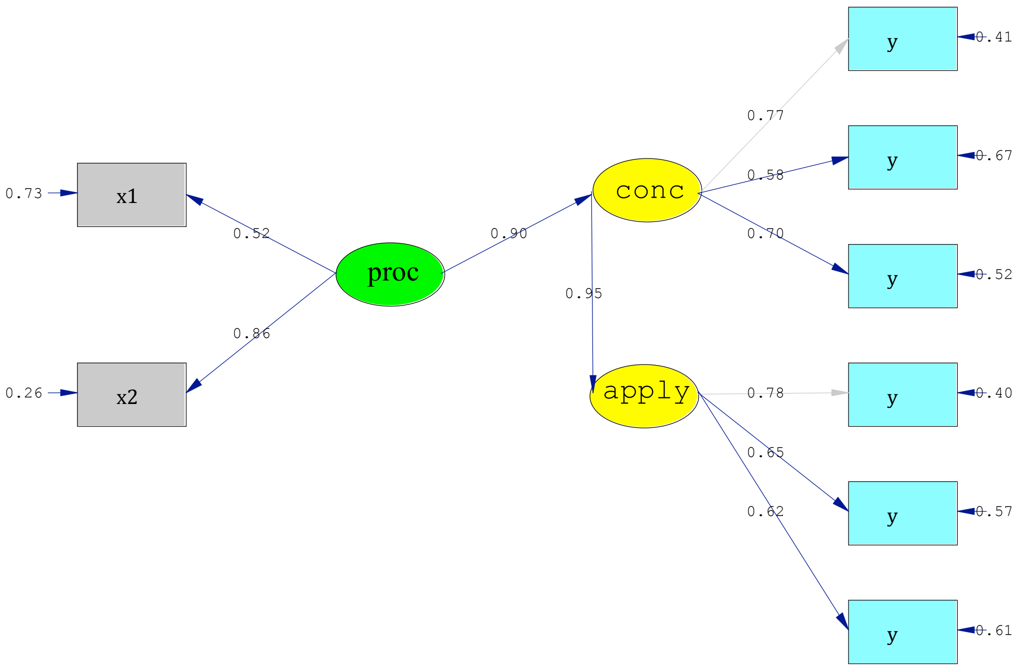


Figure 5-13. The complete model estimated in LISREL with the correlation matrix as input using maximum Likelihood estimators.

If we look at the standardized solutions[21] in Figure 5-13, we can see how each item loads on each latent variable. The factor loadings are all within the range from 0.52 to 0.86, which means that the impacts of the different items are relatively evenly distributed. However, *algebraic procedures* (x_2) have a larger impact on procedural knowledge than *graphic procedures* (x_1), with factor loadings equal to respectively 0.86 and 0.52. We can also see that *relations between graphic and algebraic representations* (y_1) has a stronger loading on conceptual knowledge of functions than *graphic interpretations* (y_2) and *algebraic interpretations* (y_3). In the same way *economic applications* (y_4) has the strongest loading on the ability to apply functions when compared to *derivation* (y_5) and *graphic knowledge of the derivative of a function* (y_6). The error term, which is displayed next to each item, is the proportion of variance that is unexplained[22].

Reliability is in general a question on how well a variable is measured, or to which degree the measure of the variable is error free (Blalock, 1982). Reliability is estimated for each item, and also for each latent variable. For an item, the standardized factor loading serves as the reliability-measure[23]:

[21] The standardized solution gives the estimates from the correlation matrix. The factor loadings are standardized within the range from -1 to 1.

[22] Error term=$1-\lambda^2$.

[23] Standardized error=1-reliability=1-(standardized loading)^2 (Hair et al., 1998).

$$\rho_i = \lambda_i^2 \qquad (5.5)$$

Table 5.19 shows the factor loadings in the model.

Table 5-19. Factor loadings from the estimated model.

λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8
0.52	0.86	0.77	0.58	0.70	0.78	0.65	0.62

The item reliabilities should exceed 0.50 which means that the standardized factor loading should exceed approximately 0.70. Table 5-19 shows that some factor loadings are below 0.70, but the results are reasonable. The reliability measure for the graphic procedures is lowest. It is impossible to determine the reasons for this, but it might partly be explained by the fact that the results are hard to score. In general, students have problems with drawing graphs, and giving a score on a partly correct graph must rely on judgment.

Reliability measures for each latent variable are not provided by LISREL, but they can easily be calculated. Since each item is supposed to load on the common factor, the item scores need to be consistent in the sense that they need to be inter-correlated to some degree. When only two items load on a latent variable, such as procedural knowledge of functions, the Pearson correlation between the items gives an indication on reliability. A very low correlation is not consistent with the fact that the items indicate the same concept (latent variable). On the other hand, a correlation close to 1 could mean that the items are in fact equal measures and that one of them could be left out.

Procedural knowledge of function was measured by *graphic procedures* (x_1) and *algebraic procedures* (x_2) with an estimated correlation of $r = 0.4492$.

The three items that are used to measure conceptual knowledge of functions are correlated with each other in the range form 0.44 to 0.52.

Table 5-20. Correlations between items that measure conceptual knowledge of functions.

	y_1	y_2	y_3
y_1	1		
y_2	0.4523	1	
y_3	0.5226	0.4437	1

The underlying assumption in the original model is that the observed variables, for example *graphic interpretations* (y_2) and *algebraic interpretations* (y_3), both correspond to the meaning of conceptual knowledge of functions. They correlate to some degree, but no correlations are larger than 0.52, as the three items do not measure exactly the same aspect of knowledge. All three inter-correlations are in the range 0.44 and 0.52, suggesting that the internal consistency is satisfying for this concept.

The same argumentation holds for the internal consistency of the items measuring the ability to apply functions as shown in Table 5-21. The correlation between *derivation* (y_5) and *graphic knowledge of the derivative of a function* (y_6) is lower than that of the others.

Table 5-21. Correlations between items that measure the ability to apply functions.

	y_4	y_5	y_6
y_4	1		
y_5	0.5374	1	
y_6	0.4587	0.3914	1

Even if both items reflect the ability to apply functions in relation to the derivative, one can say that they are very distinct in nature in the sense that grasping the meaning of a graph is something quite different than working with algebraic expressions. Nevertheless, both facets relates to application of functions.

One of the most common used measures for reliability in social sciences is Cronbach's alpha (Cronbach, 1951):

$$\alpha = \frac{k}{k-1}\left(1 - \frac{k}{k + 2\sum_{i \neq j} r_{ij}}\right) \qquad (5.6)$$

where k is the number of items loading on the latent variable and r_{ij} is the correlation between item i and item j. Hair et al. (1998, p. 612) suggests construct[24] reliability expressed by the standardized loadings and measurement errors:

$$CR = \frac{(\sum_i \lambda_i)^2}{(\sum_i \lambda_i)^2 + \sum_i Var(\varepsilon_i)} \qquad (5.7)$$

Measures for internal consistency are calculated for each item. Estimates for Cronbach's alpha[25] and construct reliability (CR) are presented in Table 5-22. These are both measures of internal consistency, based on the average inter-item correlation.

Table 5-22. Inter item correlations in terms of Cronbach's alpha and construct reliability (CR) for the latent variables.

	Procedural knowledge of functions	Conceptual knowledge of functions	Ability to apply functions
Cronbach's alpha	0.62	0.73	0.72
CR	0.66	0.74	0.73

Although there is no exact rule for acceptable values, a rule of thumb says that Cronbach's alpha and CR should be larger than 0.7, but not exceed 0.9. All the measures in Table 5-22 are within acceptable range, even if the internal consistency of procedural knowledge of functions is slightly lower than 0.7. Since only two items measure procedural knowledge of functions, the values for Cronbach's alpha are reasonable.

Reliability of measures refers to the accuracy of the measurement instruments. When we try to measure conceptual knowledge with a set of tasks as the measurement instrument, it is obvious that errors in measurements are unavoidable and that we do not get an exact score of a student's level of knowledge. What we can expect is to attach some

[24] The term construct is often used for the latent variable concept .
[25] In the SPSS output Cronbach's alpha is referred to as Standardized item alpha.

idea of this level with some uncertainty. Therefore the factor derived from the response to example items y_1, y_2 and y_3 contains one proportion which is the true level of knowledge and another proportion being the measurement error:

Variance in measure = Variance in true score + Variance in error

In general the reliability (ρ) expresses the underlying true score as a proportion of what we measure:

$$\rho = \frac{Var(T)}{Var(T) + Var(\varepsilon)} \qquad (5.8)$$

where the underlying true score is T score and ε is the error of measurement.

According to Fornell and Larker (1981), CR in formula 5.7 does not measure the amount of variance that is captured by the construct in relation to the amount of variance due to measurement error. Instead they propose a more conservative measure, the average variance extracted (ρ_{ave}), calculated as:

$$\rho_{ave} = \frac{\sum \lambda_{yi}^{2}}{\sum \lambda_{yi}^{2} + \sum Var(\varepsilon)} \qquad (5.9)$$

The table below shows the estimated values for the average variance extracted.

Table 5-23. Estimates of average variance extracted.

	Procedural knowledge of functions	Conceptual knowledge of functions	Ability to apply functions
Variance extracted	0.51	0.56	0.47

If $\rho_{ave} > 0.50$ then the variance captured by the construct is larger than the variance due to measurement error and indicates that the reliability of the construct is adequate, while the reliability is more questionable when the value falls below 0.50. The estimate of average variance extracted for procedural knowledge of function is just within the recommended range, while the measure for the ability to apply functions is slightly below. Keeping in mind that this is a conservative measure, the conclusion is that the reliabilities seem reasonably satisfying. The degree of inter-item correlation in this regard is sometimes referred to as convergent validity. It is expected that items that measure different facets of the same concept, should to some extent be correlated. Another question is to which degree the measurements of different concepts diverge from each other. The items that measure procedural knowledge of functions should to some degree diverge from the items that measure conceptual knowledge of functions. The hypothesis is that there is a relationship between procedural and conceptual knowledge of functions, and therefore some degree of inter-correlation between the items that measure the concepts is expected. The question is rather whether the degree of correlation is reasonable. The item x_1 that measures procedural knowledge of functions is correlated with r=0.280 with y_6 that measures the ability to apply functions. Even if they are correlated, this correlation is smaller than the correlation between x_1 and x_2. As a rule of one could say that the correlations between items measuring the same concept should be bigger than the correlation between items measuring different items. As Table 5-14 shows, there are numbers that might challenge the conclusions, but with some exceptions, the correlations between the items that load on the different measures are in general lower than the measures of inter-correlations per item. The correlations between items that load on procedural knowledge of functions were correlated with the items that load on conceptual knowledge of functions and are 0.41 on average. Items for the ability to apply functions had an average correlation with the items that measure procedural knowledge of functions and conceptual knowledge of function are respectively 0.40 and 0.44. Hence the items discriminate to some extent between the concepts, although not very strongly.

In summary, the reliability measures seem to be acceptable and different reliability scores were reasonably satisfactory. It is not possible to establish the cause for variation in reliability estimates by just looking at the tasks behind the items, but one can reflect on possible reasons by inspecting the nature of the questions in relation to the estimates. It might be that tasks that are difficult to evaluate, in terms of giving a score on the students' performance, weaken the reliability. An inspection of the reliability measures of x_1 and x_2 indicates that the set of tasks used to measure a student's ability to perform algebraic procedures (x_2) is more accurate than the measure of a student's ability to perform graphic procedures (x_1). One of the reasons for this might be found in the evaluation of the test responses from the students. The answers to some of the algebraic tasks can, more or less, be evaluated as right or wrong, while the evaluation of the quality of a graph will rely more on judgement. One of the graphic tasks was to draw a straight line, given the algebraic expression of a linear function. Apart from those who had it correct or completely wrong, many students got only the slope or only the intercept correct. In addition, the level of accuracy varied a lot between the students. A look at the estimated reliability for y_1, y_2 and y_3 that have been used to measure conceptual knowledge of functions shows that the value for y_2's reliability is somewhat lower than the others. Item y_2 measured students' ability to work with graphic interpretations where the students were asked to draw graphs based only on graphic representation of other functions. Again, the grading of the results relies on judgement, which might be a reason for the lack of accuracy. The estimated reliability of y_4, measuring students' abilities on problems related to economic applications, was higher than for y_5 and y_6 that were both related to applying the concept of function to the derivative. This is the only item with applications concerning subjects outside mathematics, but the answers were quite easy to evaluate.

This study is basically confirmative in its design as opposed to an exploratory approach. Consequently the construction of the tasks in the test relies on the characteristics of each measure. Many labels are used to describe different types of validity, but content validity refers to whether these characteristics are met. Content validity is concerned with whether all aspects of a concept are covered, which is a challenging problem in this study, especially when it comes to conceptual knowledge of functions. We might have a perception of what we mean when we talk about conceptual or maybe deeper understanding. It is possible that mathematicians' way of thinking about deeper understanding is reasonably consistent in the sense that they have a common imagination of what deeper understanding means. Even if such a consistency exists, there is still a challenge to agree on the set of criteria that would cover all aspects of such a concept. As an illustration one can easily imagine that different persons would give different descriptions of what they mean by the *concept environment*, even if they have a common understanding of it. Content validity is hard to prove and will be subject to judgment. The items that load on the same factor should reflect the meaning assigned to that particular concept and distinguish from the other latent variables (Geffen, 2003)[26]. In this study, the discussion of content validity will focus on the criteria for each concept. According to Bollen (1989, p. 183), the researcher must define the concept in a manner that covers all important aspects of the concept, and the judgment on content validity must rely on the researcher. The discussion in chapter 2 is concerned with whether the criteria for each measurement are adequate, while the discussion in chapter 4 is intended to explain how the tasks have the properties reflecting these criteria. Even if the tasks in this research are developed with the intention of reflecting the concepts in a way that seems good according to judgment, this can hardly be said to consist strong evidence for validity.

Another facet of validity, criteria validity, is to check how the operationalization performs against some other criteria that are supposed to be predicted by the measures. As an example, if the measure of the ability to apply functions is good, one would expect that students with high scores on this measure also perform well when they are exposed to other economic problems where mathematics plays a part. Therefore, the validity of the test was investigated by collecting a new sample of data on a new group of students using the same test as in the main test. In this post test, which is presented in chapter 5.13, the students were identified by a student identification number to make it possible to compare their test score with how they performed in other exams at an individual level.

5.10 PROCEDURAL-CONCEPTUAL RELATION

Before the model is evaluated further the hypothesized assumption that the causal direction goes from procedural knowledge of functions to conceptual knowledge of functions should be evaluated. The scores $z_i (i = 1, 2, 3)$ were computed for each student according to equations (4.16) and (4.17). For each student, scores on procedural knowledge of functions and conceptual knowledge of functions were estimated.

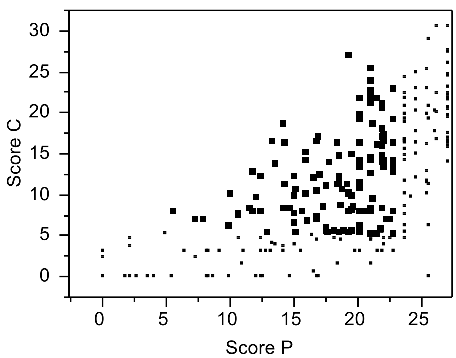


Figure 5-14. Index scores on procedural knowledge (Score P) and conceptual knowledge (Score C) of students from the main test (n=283).

26 These two aspects of validity are called convergent validity and discriminant validity (Byrne, 1998).

Figure 5-14 shows that many students score high on procedural knowledge of functions and low on conceptual knowledge of functions, while practically none of them score high on conceptual knowledge when procedural knowledge is low. The result clearly support the genetic view (Haapasalo & Kadijevich, 2000) that procedural knowledge is a necessary but not sufficient condition for conceptual knowledge, at least for these students in this context when it comes to knowledge of functions. Some students whose scores are at the diagonal in Figure 5-14 also fit the simultaneous activation view. There is no trace of the inactivation view or the dynamic activation view in the data. The conclusion is that the hypothesized direction from procedural knowledge of functions to conceptual knowledge of functions is supported and will be maintained. Consequently, the further evaluation, analysis and interpretations are based on the competing model. Comments on both models are included to indicate that the answers to the research question are not seriously affected by the choice between the two models.

5.11 OVERALL MODEL FIT

If the concepts are measured in a reasonable manner and the structure seems good, then the estimation of the total model should be able to reproduce the input matrix as described previously. How well does the observed data fit with the model generated data? This chapter refers to the analysis where the maximum likelihood estimates were estimated on the covariance matrix. Many fit measures are reported in Goodness Fit statistics in the LISREL output as shown in Figure 5-15.

If F is the fit function, F_{min} serves as a measure for S - $\Sigma(\theta)$ and several statistics are available. One of the most frequently reported is the minimum fit function[27] Chi-square (χ^2) statistics. Here:

$$\chi^2 = (n-1)\,F_{min} \qquad (5.10)$$

with the degrees of freedom:

$$df = (1/2)(p+q)(p+q-1)-t \qquad (5.11)$$

where t is the number of parameters to be estimated. With n=283 respondents in the study, the minimum fit function Chi-square is:

$$\chi^2 = (n-1)\,F_{min} = (283-1)0.0659 = 18.57 \quad (p=0.42) \qquad (5.12)$$

which is well inside the acceptable range[28] for this measure for the overall goodness of fit. Since we do not want to reject the hypothesis, H_0: $\Sigma=\Sigma(\theta)$, we want the χ^2 estimate to be small, corresponding to the reported p-value to be large, at least $p>0.05$.

[27] The minimum fit function chi square deviates slightly from normal theory weighted lest squares, which was applied for model comparison, but the choice of Chi-square estimate does not have influence on any of the conclusions.
[28] The measure indicates good fit when p<0.05.

Goodness of Fit Statistics

Degrees of Freedom = 18
Minimum Fit Function Chi-Square = 18.57 (P = 0.42)
Normal Theory Weighted Least Squares Chi-Square = 18.97 (P = 0.39)
Estimated Non-centrality Parameter (NCP) = 0.97
90 Percent Confidence Interval for NCP = (0.0 ; 15.67)

Minimum Fit Function Value = 0.066
Population Discrepancy Function Value (F0) = 0.0034
90 Percent Confidence Interval for F0 = (0.0 ; 0.056)
Root Mean Square Error of Approximation (RMSEA) = 0.014
90 Percent Confidence Interval for RMSEA = (0.0 ; 0.056)
P-Value for Test of Close Fit (RMSEA < 0.05) = 0.91

Expected Cross-Validation Index (ECVI) = 0.19
90 Percent Confidence Interval for ECVI = (0.19 ; 0.25)
ECVI for Saturated Model = 0.26
ECVI for Independence Model = 5.56

Chi-Square for Independence Model with 28 Degrees of Freedom = 1551.77
Independence AIC = 1567.77
Model AIC = 54.97
Saturated AIC = 72.00
Independence CAIC = 1604.93
Model CAIC = 138.59
Saturated CAIC = 239.24

Normed Fit Index (NFI) = 0.99
Non-Normed Fit Index (NNFI) = 1.00
Parsimony Normed Fit Index (PNFI) = 0.64
Comparative Fit Index (CFI) = 1.00
Incremental Fit Index (IFI) = 1.00
Relative Fit Index (RFI) = 0.98

Critical N (CN) = 529.68

Root Mean Square Residual (RMR) = 0.73
Standardized RMR = 0.025

Goodness of Fit Index (GFI) = 0.98
Adjusted Goodness of Fit Index (AGFI) = 0.97
Parsimony Goodness of Fit Index (PGFI) = 0.49

Figure 5-15. Goodness of fit statistics for the complete model estimated in LISREL with the covariance matrix as input using maximum Likelihood estimators.

With large samples, it is quite common to find a large χ^2 relative to degrees of freedom, indicating a need to modify the model in order to fit the data (K. Jöreskog & Sörbom, 1993). In general, if the sample size is large, it can be expected that models that approximate the data closely will be rejected (Browne & Cudeck, 1993). It is obvious in equation (5.10) that χ^2 increase with n. Equation (5.11) makes it clear that the degrees of freedom will decrease as the number of parameters (t) increases, thereby making it more difficult to reject H_0, i.e. more likely to claim that the fit is good. Of course, introducing parameters in the model just to improve the goodness of fit should be avoided.

Another fit measure that according to Steiger (1990, p. 177) is a "natural measure of badness-of-fit of a covariance structure model" is the noncentrality parameter (NCP), denoted as λ, reported by LISREL[29]:

$$NCP=\lambda = Max(0,\chi^2 - df) \qquad (5.13)$$

where a small value indicates a good fit. A 90% confidence interval for λ is also given to estimate precision. This gives an estimate for the Non Centrality Parameter, NCP:

$$NCP = Max(0, 18.97 - 18) = 0.97 \qquad (5.14)$$

There are no accepted values for threshold values, but NCP is used for the comparison of models. The value is slightly lower than for the original model (NCP=1.48), which indicates that removing the link between procedural knowledge of functions and the ability to apply functions gives a moderately better fit.

To account for the problem that models that hold approximately in the population will be rejected in large samples, Brown and Cudec (1993) proposed the population discrepancy function (PDF) (McDonald, 1989). PDF is a fit measure that takes into account the error of approximation in the population, and is defined as:

$$PDF = Max\left(F_{\min} - \frac{df}{n-1}, 0\right) \qquad (5.15)$$

The Population Discrepancy Function (PDF) is estimated to:

$$PDF = Max\left(0.0673 - (18/282), 0\right) = 0.0034 \qquad (5.16)$$

The corresponding estimate for the original model is 0.0053, which, again, justifies the modification to the model. A problem with PDF is that it decreases as parameters are added to the model, thus indicating a better fit. One of the most commonly used goodness of fit measures, where this problem is in an adjusted form, is the Root Mean Square Error of Approximation (RMSEA), first proposed by Steiger and Lind (1980), and defined as:

$$RMSEA = \sqrt{\frac{PDF}{df}} \qquad (5.17)$$

[29] The minimum fit Chi-square is used.

The estimate for RMSEA is:

$$RMSEA = \sqrt{\frac{PDF}{df}} = \sqrt{\frac{0.0034}{18}} = 0.014 \tag{5.18}$$

RMSEA is an indication on "How well would the model, with unknown, but optimally chosen parameter values, fit the population covariance matrix if it were available?" (Browne & Cudeck, 1993, pp. 137-138). Values lower than 0.05 indicate good fit, while values higher than 0.10 indicate poor fit. RMSEA was estimated as 0.018 for the original model, favouring the adjustments. LISREL reports a 90% confidence interval for RMSEA, giving the researcher an indication of how precise the measure is. LISREL also reports a test value for the goodness of close fit, testing whether RMSEA is lower than 0.05. Jöreskog and Sörbom (1996) have suggested that the reported p-value should be larger than 0.05. The output reports a p-value equal to 0.91, which is very convincing.

Another class of fit indices, with values normally between 0 and 1, and measures how much better the model fits when compared to a baseline model. The latter is often the so-called independence model, i.e. a model in which the correlations between all the variables are zero. One of these indices is the Normed Fit Index, NFI (Bentler & Bonett, 1980). If F is the fit function and F_i is the fit function for the independence model, then

$$NFI = 1 - \frac{F_{\min}}{F_{i,\min}} \tag{5.19}$$

A value larger than 0.90 indicates acceptable fit to the data (Byrne, 1998).

The LISREL output reports Normed Fit index NFI = 0.99. The value is in the range from -1 to 1 and a common recommendation is that the value should be bigger than 0.90. The corresponding value for the original model is 0.98.

Many estimation methods and fit measures, in addition to those mentioned here, have been developed, and several fit measures and statistics are reported by LISREL. The measures reported above point to the direction of support for the model, and the synthesis of all these estimates is a strengthened conclusion.

5.12 SUMMARY OF ANALYSIS OF THE MAIN MODEL

To summarize this chapter, the statistical results are satisfactory, in the sense that the estimated parameters and measures for fit and reliability are within the acceptable range, and kurtosis is not likely to disturb the conclusions. Alternative estimation methods gave only marginal changes to the estimates, which supports the model. The estimates favoured the adjustment made to the model. We have seen that the estimated parameters and fit measures did not seem to be very sensitive to neither the estimation method nor the type of matrix, which in general gives support for the robustness of the model.

The latent variable model estimation proved that conceptual knowledge of functions depends on procedural knowledge of functions. The ability to apply functions depends significantly on conceptual knowledge of functions. When intermediated by conceptual knowledge, procedural knowledge clearly affects the ability to apply functions. In other words, procedural knowledge plays an important role for conceptual knowledge, which, in turn is a necessary condition for the ability to apply functions, while procedural knowledge of functions alone seems to be insufficient as a platform for application.

5.13 POST TEST

The average total score on the post test is lower than the average score from the students in the main test with a mean score of 45.9 compared to 75.6 in the main test. The post test class had higher percentage of students that did not pass the multiple-choice exam in mathematics. The same tendency could be seen in the exam in business economics. Despite the low average scores, there is a certain variation in scores within the class with a standard deviation of 22.9, and it is therefore possible to distinguish between students. The discussion of validity concerns the battery of questions and the way they are measured rather than the subjects exposed to them. It is the variance between the students' performances that provides us with information in this regard.

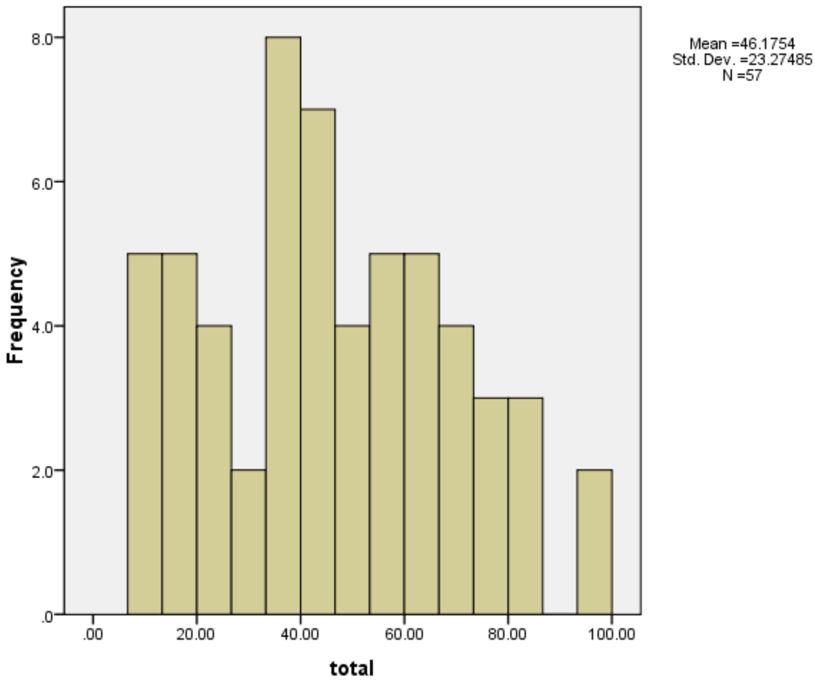


Figure 5-16. Distribution of total scores from the post test.

In addition, the factor loadings from the final model were used to compute three index scores, one for each latent variable. The summary statistics of the index score for the three measurements are shown in Table 5-24.

Table 5-24. Statistics for location and dispersion of index scores.

Index scores from post test

	N	Minimum	Maximum	Mean	Std.Dev.
Index score on procedural knowledge of functions	57	1.04	24.44	12.84	5.82
Index score on conceptual knowledge of functions	57	0.00	20.06	7.93	5.30
Index score on the ability to apply functions	57	0.00	31.73	12.75	9.05

Since the scale of the scores is different, it does not make any sense to compare the mean scores.The important observation in Table 5-24 is that the standard deviation indicates a clear variation between the students, which enables us to distinguish between them. The question is how these scores compare with the outcome in the exams. The following tasks from the math test were used to indicate procedural performance in mathematics:

Problem 1

Compute the value of $\left(x^3 - 2y^2\right)^2$ when $x = -1$ and $y = 2$. The answer is:

A 27
B -27
C 49
D 81
E I prefer not to answer the question

Problem 2

Perform the polynomial division: $(x^2 - 6x + 10) : (x - 3)$ The answer is:

A $x - 3 + \frac{1}{x - 3}$
B $x^2 + 3$
C $x - 3$
D $x^2 + 3 + \frac{1}{x + 3}$
E I prefer not to answer the question

Problem 3

A straight line passes through the points $(x_1, y_1) = (1, 4)$ and $(x_2, y_2) = (4, 10)$.
The equation for this line is given by:

A $y = -x - 1$
B $y = -2x - 2$
C $y = 2x + 2$
D $y = -x + 4$
E I prefer not to answer the question

All these tasks are skill oriented in the sense that it should be possible for a student who masters the algorithms involved to achieve the correct answer. Answering questions in multiple choice exams can sometimes be approached by testing the solutions. In problem 1, the negative answer should be easy to eliminate, since the algebraic expression is squared. Despite that, it is reasonable to believe that students do the computation with the three alternatives left to choose from. A similar approach is possible in problem 2, by for example finding the numerator in the remainder first, but there are still two alternatives to choose between. In problem 3, it should be easy to verify that the two points that are given fit into the line. Altogether, the solution in all three cases only requires that an algorithm must be used. Exam performance in procedural tasks was defined as the number of correct answers on a scale from 0 to 3. The table below compares exam performance with the index scores.

Table 5-25. Mean scores on exam performance in procedural tasks by index scores.

Index scores compared to exam performance in procedural tasks

		Index score on procedural knowledge of functions	Index score on conceptual knowledge of functions	Index score on the ability to apply functions
		Mean	Mean	Mean
Exam performance in procedural tasks. The number of students in parenthesis.	0 (2)	7.1	3.1	0.0
	1 (20)	10.2	5.9	10.6
	2 (25)	13.7	8.9	13.5
	3 (10)	17.2	10.5	17.6

Low scores on the indexes correspond to low scores on the exam performances. This is not surprising since the high performers among the students score higher on all scales. An interesting observation is that the mean index score on procedural knowledge predicts the score on the exam in a linear pattern, as the numbers in the first column increase by 3.1, 3.5 and 3.5 respectively from one category to the next. One should be careful to draw certain conclusions on the basis of a relatively small sample, but the pattern is that the index score on procedural knowledge of functions predicts the performances in procedural tasks to some degree. The two other index scores, and especially the score on the ability to apply functions, shows a slightly less regular pattern. Estimates of Pearson's correlation coefficients gave a slightly higher estimate for the association between exam performance in procedural tasks and the index score of procedural knowledge of functions (r=0.47) than with the two other index scores (r = 0.36 for both). The following tasks from the math test were used to indicate conceptual performance.

Problem 4

Look at the function $f(x) = x^2 + 2x + 4$, $D_f = R$. The following is true:

A The function has only positive values
B The function has both positive and negative values
C The function has only negative values
D None of the answers above are correct
E I prefer not to answer the question

Problem 5

Look at the function $f(x) = x^2 + 2x + 4$, $D_f = R$. The following is true:

A The function has its maximum value for $x = -1$
B The function has its maximum value for $x = -2$
C The function has its minimum value for $x = -1$
D The function has its minimum value for $x = -2$
E I prefer not to answer the question

Problem 6

Which of the following is true?

A The sum of two linear functions (of first degree) is a quadratic function (of second degree)
B The sum of two linear functions (of first degree) is a cubic function (of third degree)
C The product of two linear functions (of first degree) is a linear function (of first degree)
D The product of two linear functions (of first degree) is a quadratic function (of second degree)
E I prefer not to answer the question

All problems require that the students select an appropriate solution strategy based on their understanding of the question. A mathematician would most likely have an image of the graph of the function in mind related to problems 4 and 5, but it is also possible to apply an algebraic approach to these problems. Several solution strategies are applicable, but they require that pieces of knowledge are combined appropriately. In problem 6, the question and all the answers are presented by text that must be interpreted in a mathematical context. Exam performance in conceptual tasks was defined as the number of correct answers on a scale from 0 to 3. The table below compares exam performance with the index scores.

Table 5-26. Mean scores on exam performance in conceptual tasks by index scores.

Index scores compared to exam performance in conceptual tasks

		Index score on procedural knowledge of functions	Index score on conceptual knowledge of functions	Index score on the ability to apply functions
		Mean	Mean	Mean
Exam performance in conceptual tasks. The number of students in parenthesis.	0 (5)	12.6	3.7	5.8
	1 (17)	9.5	6.9	11.7
	2 (17)	14.1	7.6	13.5
	3 (18)	14.9	10.4	15.0

Again, a low score on the indexes corresponds to a low score on the performances on the exam and the index score on conceptual knowledge predicts the score on the exam in a more regular linear pattern than the other two indexes. Also, the estimates of Pearson's correlation coefficients gave a slightly higher estimate for the association between exam performance in conceptual tasks and the index score conceptual knowledge of functions (r=0.36) than with the two other index scores (r = 0.30 and r = 0.36 respectively).

The marks from the exam in business economics were used to indicate performance in business economics. The marks are scaled according to the ECTS system where A is the best. The mark F means failure.

Table 5-27. Marks on exam performance in business economics by index scores.

Index scores compared to exam performance in business economics

		Index score on procedural knowledge of functions	Index score on conceptual knowledge of functions	Index score on the ability to apply functions
		Mean	Mean	Mean
Exam performance in business economics. The number of students in parenthesis.	A (1)	12.4	17.7	31.7
	B (3)	22.6	16.5	22.2
	C (7)	13.1	7.5	14.4
	D (17)	13.3	9.2	14.4
	E (11)	13.4	6.0	12.3
	F (18)	10.4	6.1	8.2

One can see that the index score on the ability to apply functions to some degree seems to predict the exam performance in business economics slightly better than the others. One can also see that the index score on procedural knowledge of functions was between 10.4 and 13.4 for all student categories except for the category that achieved mark B, while the index score on conceptual knowledge of functions only separated the As and Bs from the others.

Since the marks are at ordinal level, Spearman's correlation coefficients were estimated to indicate the degree of association between exam performance in business economics and the three index scores. The correlation against the index score on the ability to apply functions is highest (r = 0.40) than the two others (r = 0.32 and r = 0.35 respectively).

Research questions 2 and 3 are about relationships, and the results from the post test were investigated according to the findings of this study by a set of linear regression equations and a test for mediation effects (Sobel, 1982). The index scores were applied for this purpose, and the following variables were defined:

V_1 = Index score on procedural knowledge of functions
V_2 = Index score on conceptual knowledge of functions
V_3 = Index score on the ability to apply functions

The index score on conceptual knowledge of functions (V_2) increased in a reasonably linear pattern with the index score on procedural knowledge of functions (V_1) as seen from Figure 5-17. The pattern does support the genetic view and the simultaneous activation view, but it is slightly distinguished from the pattern in Figure 5.14, as a few students seem to score high on conceptual knowledge of functions and low on procedural knowledge of functions.

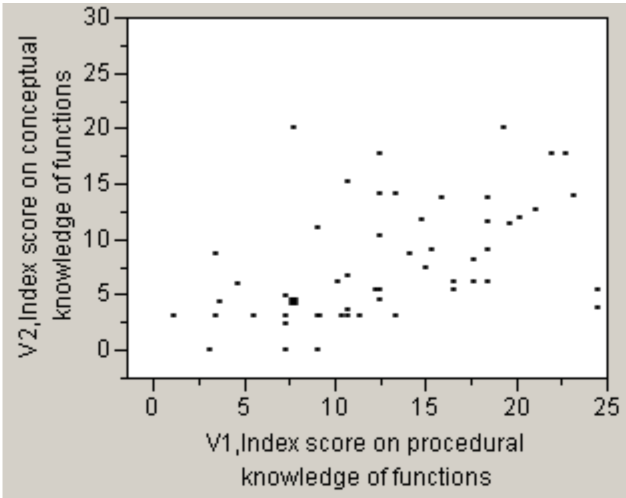


Figure 5-17. Relationship between V_1 and V_2.

Equation (5.20) shows that V_1 is a significant explanatory variable for V_2, which corresponds to the conclusion on research question 2: conceptual knowledge of functions depends on procedural knowledge of functions.

$$V_2 = 2.53 + \underset{\substack{(p=0.0003) \\ t=3.86}}{0.42} \cdot V_1 \qquad (5.20)$$

Research question number 3, addressing how the ability to apply functions depends on the two other types of knowledge, was also investigated. First, the index score on conceptual knowledge of functions (V_2) was used as an independent variable with the index score on the ability to apply functions (V_3) as dependent. Figure 5.18 shows a linear pattern between these variables and equation (5.21) shows that V_2 has a significant effect at any reasonable level ($p<0.0001$) when V_2 is the only independent variable.

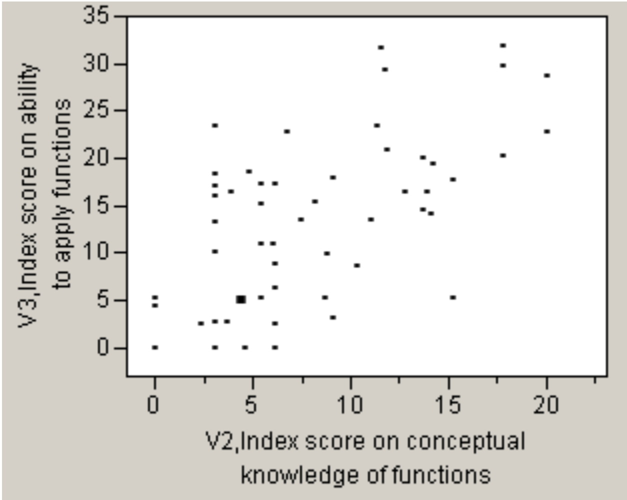


Figure 5-18. Relationship between V2 and V3.

$$V_3 = 4.31 + 1.06 \cdot V_2 \qquad (5.21)$$
$$(p<0.0001)$$
$$t=5.92$$

Again, this result is not surprising, but it does not fully cover the conclusions on research question 3 since effects from procedural knowledge of functions are not included. Since we have a set of structural equations where V_2 is a function of V_1 (equation (5.20)), one should be cautious to interpret the regression parameter in equation (5.21). Equations (5.20) and (5.21) are a set of simultaneous equations. Therefore, the effect from V_2 on V_3 was also estimated by a two-stage least square approach (Studenmund, 2001) with V_3 as the dependent variable with the predicted values for V_2 from equation (5.20) as independent.

$$V_3 = -1.11 + 1.75 \cdot \hat{V}_2 \qquad (5.22)$$
$$(p=0.0002)$$
$$t=3.98$$

When the two-stage least squares procedure is applied, the effect from the index score on conceptual knowledge of functions is still clearly significant as seen in equation (5.22).

The partial plot of the index score on procedural knowledge of functions (V_1) against the index score of the ability to apply functions (V_3) is reasonably linear as seen in Figure 5.19.

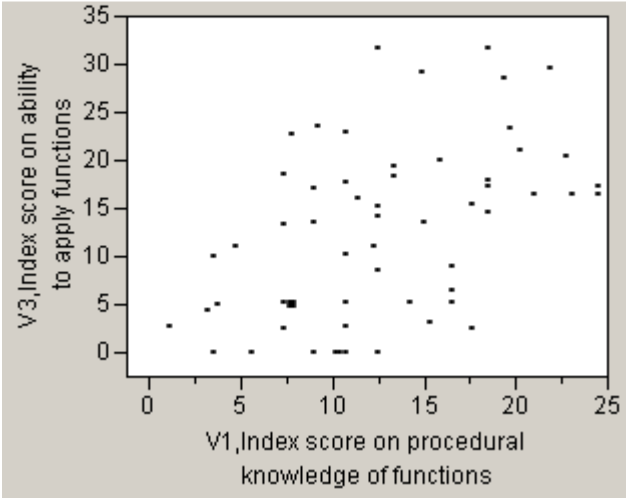


Figure 5-19. Relationship between V_1 and V_3.

The regression equation with V_1 as the only explanatory variable is:

$$V3 = 3.30 + 0.74 \cdot V1 \qquad (5.23)$$
$$(p=0.0002)$$
$$t=3.98$$

However, the relationship between V_1 and V_3 may be fully or partly mediated by V_2. One conclusion from the main study is that procedural knowledge of functions seems to have an effect on the ability to apply functions mediated by conceptual knowledge of functions. Therefore, it is interesting to see if the same holds when the index scores are applied. According to Preacher and Leonardelli (2006), four conditions must be present for V_2 to be a mediator:

1 V_1 must have a significant effect on V_2
2 V_1 must have a significant effect on V_3 when V_2 is omitted
3 V_2 must have a significant unique effect on V_3
4 The effect of V_1 on V_3 shrinks upon the addition of V_2 to the model

Conditions 1, 2 and 3 are satisfied according to the linear equations discussed so far. A linear regression equation with V_3 as the dependent variable with V_1 and V_2 as independent is estimated.

$$V_3 = 1.08 + 0.37 \cdot V_1 + 0.88 \cdot V_2 \qquad (5.24)$$
$$(p=0.0468) \quad (p<0.0001)$$
$$t=2.03 \qquad t=4.46$$

Equations (5.2)3 and (5.24) show that the effect of V_1 on V_3 has decreased when V_2 is included. Hence condition 4 seems to be satisfied. According to Preacher and Leonardelli (2006), these criteria can only be used to judge informally whether mediation is occurring.

Instead, a more formal test suggested by Sobel (1982) is applied to test for mediation. A two-tailed z-test of the null hypothesis that the mediated effect equals zero against the alternative that mediation exists was used (Preacher & Leonardelli, 2006).

$$z = \frac{a \cdot b}{\sqrt{b^2 \cdot s_a^2 + a^2 \cdot s_b^2}} \qquad (5.25)$$

The Sobel test z-value is estimated by equation (5.25) where $a = 0.4209$ and $s_a = 0.1090$ are the regression coefficient and standard error of equation (5.20) to the parameter V_1, while $b = 0.8796$ and $s_b = 0.1972$ are the regression coefficient and standard error of the regression coefficient to V_2 in equation (5.24). The estimated z-value is $z = 2.92$ with $p = 0.0035$ and the mediating effect from V_2 is clearly significant. One could claim that the estimates for b are unreliable due to a violation of the classical assumption in regression analysis that all explanatory variables should be uncorrelated with the error term. Substituting b with the corresponding parameter estimates and standard error from two-stage least square approach in equation (5.22) gave a z-value of 2.77 ($b = 1.7486$ and $s_b = 0.4391$), which is still significant at any reasonable level ($p = 0.0056$).

The conclusion from Sobel's test shows that the effect from V_1 on V_3 is clearly mediated by V_2. This is in accordance with the earlier conclusions on research question 3, that there is an indirect effect from procedural knowledge of functions on the ability to apply functions through conceptual knowledge of functions. However, the test does not say anything about the absence of a direct effect from procedural knowledge of functions on the ability to apply functions, but it is the presence of a significant mediating effect that is the most important conclusion.

5.14 INTERVIEW

Emma, Anna and Martin were interviewed to form a quasi peer group, all with background at upper secondary schools in Norway.

When asking Emma about her view of mathematics, she said that she did not think that mathematics would be important in her professional career. She said she had actually never liked mathematics, but she felt motivated when she managed to solve problems. Her primary motivation was to complete the course, meaning, to pass the exams. She found it hard to describe what she meant by understanding mathematics. When posing the above-mentioned question about equations, Emma answered:

– Rules.

When asked to think about what the equation means or why it is there, Emma replied:

– No, I do not. We're supposed to find x.

Anna described her reason for learning as something that is needed to pass the exam. She said that she found it hard to relate mathematics to practical problems, even though she realizes that mathematics is related to economics. When asking what she meant by learning mathematics, she struggled to answer, so the question was modified to what she meant by understanding for example equations. Anna answered:

– To solve equations.

When asking if she thought mathematics could be used to represent something or be a model for something in the real world, she said that she could not answer.

Martin thinks mathematics is fun and he is fascinated by all the applications of mathematics. He mentioned examples from physics, economics and other areas where mathematics can be used to study many phenomena. When asked what does it mean to understand something mathematically, he answered:

– When you can use it. When you can see a problem, that is: here is a problem, how can I solve it? Then maybe you find out that differentiation can contribute to solve the problem.

To assess how the interviewees focus their concentration in a learning situation, all three were asked to comment on how they recognize themselves in the four points listed below:

1 How problems are solved
2 Why is this important?
3 How can you relate this to other knowledge, as for example other subjects?
4 Remember rules

Emma's answer was:

– It is the first and the last. Definitely not number three. I am not there, yet.

To follow up her comment on not being "there yet", she was asked if she thought it would be easier if relationships to other subjects were addressed before learning rules. She said she would prefer learning mathematics first before looking for relationships to economics, for example:

– Business economics is more advanced. You go much deeper into things. It might be all right to use your brain a little bit, at first.

When repeatedly asked if she would have mathematics at first and then economics, or vice versa she replied:

– Mathematics first, then economics.

Anna's immediate response to how she recognized herself in the four points mentioned above was:

– Yes, I would say the first one.

She mentioned the exams as the reason for why she thinks it is important to learn mathematics. She made only few remarks when asked about relationships between mathematics and other subjects:

– It depends on how deep you go into relationships. If you learn about relationships first, you must take only surface properties. Then you begin to calculate and then you maybe understand more at a later point in time. It depends on how complex it is.

She does not concentrate on remembering rules, because the exam is an open book exam, but said:

-Except such simple rules you must know to remember how you shall do it.

Martin was also asked how he focused his attention by giving the four points. He gave a reflection on all four:

– How a problem is solved is where you want to end. The reason why it is important has more to do with motivation. If I cannot explain it, in one way or another, why it is important, then I don't bother. Then I don't manage to understand it because you just sit there wondering: What do I need that for? Relationships help to remember things like when you get a deeper understanding of it. Rules and formulas have actually come by themselves for my part. So there has never been speaking of sitting down and learning formulas by rote learning. It might be because the teacher we had at the upper secondary school emphasized to show calculations. You should show the formulas that you use. That way, I have repeated the formulas I had at the upper secondary school many times.

Table 5-28. Summary of beliefs among peer group.

Name	**Beliefs**	**Questions**	**Answers and quotes**
Emma	Procedural focus	How Why Relationships Rules	"It is the first and the last. Definitely not number three. I'm not there yet." Prefer to learn mathematics first before looking at relationships to economics
	Procedural view	What is learning mathematics?	"To solve equations" "We're supposed to find x"
	Strategic motivation	Motivation for learning	To pass the exam. Does not think it is relevant for her future profession
Anna	Procedural focus	How Why Relationships Rules	"I would say the first one, to the most extent" "You must know to remember how you shall do it" Struggles to find relationships Does not focus on remembering rules
	Procedural view	What is learning mathematics?	"To solve equations" Wants to learn mathematics before applying it in economics
	Strategic motivation	Motivation for learning	To pass the exam. Finds it hard to relate mathematics to practical problems. Realizes that mathematics is related to economics
Martin	Procedural and Conceptual focus	How Why Relationships Rules	How a problem I solved is where you want to end Why is important Relationships help to remember things Rules and formulas have actually come by itself
	Conceptual view	What is learning mathematics?	"When you can use it. When you can see a problem, that is: here is a problem, how can I solve it?"
	Motivated by application and strategic motivation	Motivation for learning	Fascinated by applications of mathematics

One might get the impression that Emma was not consistent, or found the questions difficult to answer. Like Emma, Anna gave the impression as someone with focus on learning procedures. Anna admits that relationships might be important, because if she does not see the relationships, then she tends to become frustrated. This is related to Martin's argument that the reason for something to be important is motivation and he believes that seeing relationships help to remember things.

Table 5.28 summarizes some of the beliefs among the peer group. Emma's and Anna's beliefs about mathematics seemed to be in contrast with Martin's ones. They both struggled to find the words to explain what it means to understand something mathematically, but gave quite similar answers. Their motivation was to be able to pass the exam. This is in coherence with what Biggs (1993) calls the 'strategic approach to learning'. He argues for the existence of strategic approach to learning, in addition to the 'deep approach' and the 'surface approach'. Strategic approach to learning refers to an intention to achieve the best grades possible by adopting the assessment demands. These three approaches to learning do not exclude each other. It is rather the combination of them that determines the student's approaches to learning (Entwistle 1988).

Using Bigg's terminology, Emma and Anna seem to have a combination of a strategic and surface approach to learning. Martin also mentioned that he was motivated by assessment, but with a substantial element of deep approach to learning.

All three students were asked how they solved tasks in the main test and how they described their solution process. The tasks used in the interview are shown in Table 5-29.

When Emma was asked to solve tasks that required procedural knowledge in the main test, she struggled even at the easiest problems. In task 3, she started out by writing - x – 3=0, but ran into problems when she got -x=3. She seemed to search for rules like "can x be minus" or whether it is allowed to switch signs. When she was asked about the two first questions in task 4, she had great problems with basic calculations such as eight multiplied by eight and 64-48, when she applied the formula for finding the zero points. She had no idea of how to do to draw the graph of f(x).

Despite serious problems with procedural tasks, she immediately gave a correct answer to one of the conceptual tasks, task 6:

– Isn't it just f(x) = 4?

Even if Emma managed to solve task 6, she said that she focused on remembering procedures. When I asked her why, she replied:

– Yes, today I do that because I think it is so difficult.

Anna had no problems with calculating function values in task 4. She hesitated for a moment before she suggested the solution to task 6. She answered:

– I think it is four.

I asked her if she thought that task 4 was easier than number 6:

– Yes, actually. It is really easy. If I am allowed to put in numbers, then I am happy.

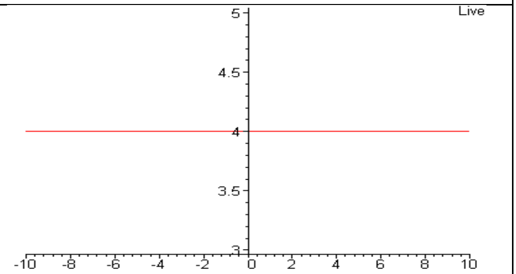
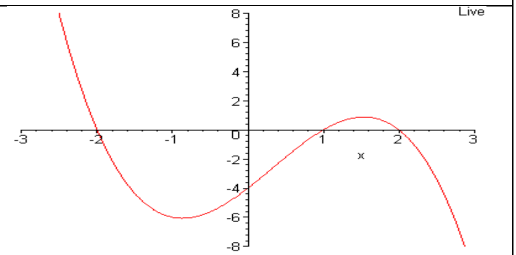
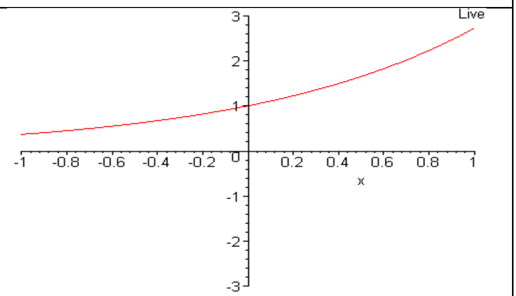
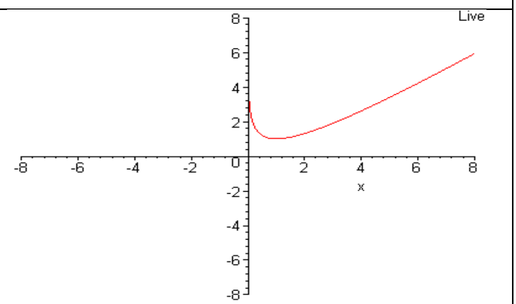
Task 12 was about adding two functions represented by their graphs. Her immediate strategy was a procedural approach to solve the problem. She found the algebraic expression for each of the two functions, added them and then sketched the graph, which she did right. Then she was asked about task 11. I asked her what –f(x) looks like. After some hesitation, she said:

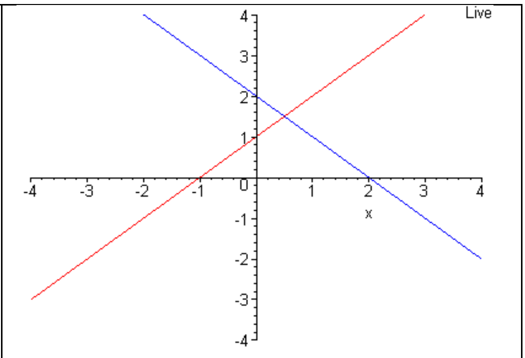
– Isn't it just the opposite?

When asked what she meant by the opposite, she answered:

– *I thought like this* (drawing the graph in the third quadrant), *not like this* (drawing the graph correct).

Table 5-29. A task for stimulated recall used in interview.

Task 3	Given f(x) = - x – 3. For which value of x is f(x) = 0 ?	
Task 4	In this exercise we look at the function $f(x) = 2x^2 - 8x + 6$, $D_f = R$ Calculate f(x) when x = -1 and when x = 4 When is f(x)=0 ? When is f(x)$\leq$0 ?	
Task 6	The graph of f(x) is shown. Write down the expression for f(x).	
Task 8	A function of third degree has the form $f(x)=ax^3+ bx^2+cx+d$. The graph of f(x) is sketched. Find d.	
Task 10	The graph of f(x) is shown. Sketch the graph of f(-x). You don't need to put more numbers on the axis. A rough sketch is enough.	
Task 11	The graph of f(x) is shown. Sketch the graph of -f(x). You don't need to put more numbers on the axis. A rough sketch is enough.	

Task 12	The graphs of two functions are show. Sketch the graph of the sum of the two functions. You don't need to put more numbers on the axis. A rough sketch is enough.	

The next question was about task 10. Anna's immediate comment was:

– f(-x)? I feel like calculating it.

After a short while she sketched it right.

Martin had no problems with answering the procedurally tasks such as task 4 and had no problem with the algebraic expressions. He also answered task 8 correctly without hesitation.

When he was asked about task 11, he was uncertain when asked about –f(x):

– Around the x-axis, isn't it? In similar way? It goes.., either like this or like this or it is? Yes it is. Either that or around the x-axis, I do not manage to imagine how it can be turned around the x-axis.

Then he drew the graph correct, but was still uncertain if it was right. When I asked him about task 10, his immediate response was:

– Isn't that the graph of e^x ?

After assuming that he sketched the graph of e^{-x}.

Table 5-30 gives a summary of the interviewees' responses to the stimulated recall tasks with respect to procedural and conceptual knowledge. Emma, who was struggling most, said she looked for procedures, but uncovered serious problems with simple algebra. The only task she solved without hesitation was the problem in task 6. Anna's responses to tasks 4 and her comment to task 10 that "I feel like calculating it" remind of a procedural oriented learner. However, she managed to solve task 10 on her own, which requires conceptual knowledge. Anna and Emma both seemed to look for procedures, but their performances pointed in a somewhat different direction.

Martin appears to be a student with algebraic skills, as he gives an impression of looking for relationships and applications. Despite this, he still struggled with some of the procedural tasks and, as with task 10, he looked for an algebraic way out of the problem. He even was aware that his teacher form upper secondary school had influenced his focus on formulas. Despite seeing himself as a learner looking for applications and relationships, he appeared to struggle with conceptual tasks where graphic representations are involved.

Table 5-30. Another task for stimulated recall used in interview.

Name	Procedural knowledge	Conceptual knowledge	Examples of thinking by stimulated recall
Emma	Struggled with basic algebraic procedures		Task 3: Modifying - x – 3=0 to -x=3, but being unable to continue Task 4: problems with basic calculations like eight multiplied by eight
	Struggled with graphic procedures	Problem with relationship of representations	Task 4: Unable to draw the graph of $f(x) = 2x^2 - 8x + 6$
		Recognized relationship between representations	Task 6: Found *f(x)* = 4 from graph
Anna	Succeeded with simple algebraic procedures		Task4: Calculated function values for $f(x) = 2x^2 - 8x + 6$ when $x = -1$ and $x = 4$ with some small hesitations
		Recognized relationship between representations	Task 6: Found *f(x)* = 4 from graph
	Succeeded with simple algebraic procedures	Recognized relationship between representations	Task 12: Sketched the sum of two linear functions represented graphically via algebraic procedures
		Solved tasks applying non-procedural strategy	Tasks 10 and 11: Sketched –*f(x)* and *f(-x)* right, after hesitating a while
Martin	Succeeded with simple algebraic procedures		Task4: Calculated function values for $f(x) = 2x^2 - 8x + 6$ when $x = -1$ and $x = 4$ without any problems
		Recognized relationship between representations	Task 8: Found d in $f(x)=ax^3+ bx^2+cx+d$ from graph without hesitation
		Problems with nonprocedural solution strategy. Struggled to judge on the outcome	Tasks 10 and 11: Sketched –*f(x)* and *f(-x)* right, after hesitating a while. Was uncertain whether the answer was correct. Used procedural approach on *f(-x).*

The next part of the interviews was about the students' experiences from upper secondary school.

Emma's first comment when she was asked if she could describe a typical mathematics lesson at the upper secondary school was:

– Yes, we actually worked on our own.

She confirmed that they were working with exercises, but when she was asked about the teacher, she said:

– Yeah, maybe he explained a bit, but if we didn't follow, it was in a way impossible.

She described a typical lesson where the teacher explained an example at the blackboard, and the students were asked to do similar examples afterwards. The teacher did not follow up the pupils individually and never controlled their homework. She described the learning environment as completely chaotic. The following statement is Emma's description of the situation:

– I remember him doing things at the blackboard for about five minutes, and that was it. It was like completely unserious. There were three math classes in the same room. It was not good. And everyone in my year had math at the same time of the day. At that school there were only group rooms and large assembly halls, so we just sat where we wanted to and joked and stuff.

Emma told that each class had about 30 students and that there were three classes having math at the same time. With 90 pupils spread around, she described the situation as chaotic and that many pupils, including her, often skipped the classes in mathematics.

Anna's experiences at the upper secondary school were in many regards similar. She described the learning environment at the upper secondary school as barely motivating and unserious. When she was asked about the teaching in mathematics, she said:

– It was ordinary teaching at the blackboard. He went through something at the blackboard, and then we should try, completely normal.

When asked about homework, her answer was:

– Yes, it was homework, but I never did it.

She was asked if it was followed up or controlled.

– No, never that I can remember.

She could not remember that her teacher related mathematics to other subjects, but now she started to see relationships between mathematics and economics.

Martin was asked about his experiences in the maths class at the upper secondary school. To the question on how a typical math lesson was and what the teacher did, he replied:

– It was much the teacher standing at the blackboard and showing mathematics. There was relatively little interaction with the class actually.

He said that the teacher did not follow up or speak to each student individually and that he did little to differentiate teaching between the mathematically skilled and those who struggled. In the beginning of the interview, Martin told that he was motivated by the applications of mathematics and mentioned the relationship between mathematics and physics. Martin had the same teacher in mathematics and physics at the upper secondary school. I asked him about how his teacher emphasized relationships between mathematics and physics:

– It was taught a lot of routines, it was. There is a link between math and physics, but part of the problem was that even if you were required to have math to have physics, the teaching schedules were not planned parallel, so when you came to the point where you needed derivation in physics, you hadn't necessarily learned it in mathematics…. There was no direct link that was easy to follow.

He described the teacher as someone who was very focused on teaching routines and remembering formulas and said:

– Why things are important was never actually a question.

He said that they had the calculator for that purpose, and he could not remember that they ever drew graphs manually. He repeatedly said that they didn't have to, since they had the calculator to do that. Martin was asked what kind of challenges the teacher gave to the cleverer students. He said:

– That was mainly me, and it was mainly just to continue to do the exercises from the book.

He was asked if the teacher followed up afterwards.

– Not very much. …

All three reported similar stories regarding a typical mathematics lesson at the upper secondary school, where the teachers demonstrated procedures at the blackboard and the students were given similar problems. Other activities were absent. All of them also told that there was almost no follow up of homework and very little effort from the teachers to differentiate the teaching. Especially two of the students who both struggled with mathematics, reported that the learning environment was very chaotic, and they did not feel that the teaching process was taken seriously.

6 Conclusions

The statistical analyses indicate that the tasks provide a satisfactory measurement tool for the concepts with respect to validity and reliability. The analysis confirmed a strong relationship between procedural and conceptual knowledge of functions and supports the view that procedural knowledge of functions is a necessary, but not sufficient condition for conceptual knowledge of functions. The estimation of the model revealed no significant direct effect from procedural knowledge of functions on the ability to apply functions, but when intermediated with conceptual knowledge of functions, the effect was significant. In other words, procedural knowledge alone seems to be insufficient to be able to apply functions.

The analysis concludes in favour of the genetic view and the simultaneous activation view for the study populations regarding knowledge of functions. As pointed out by Haapasalo and Kadijevich (2000), the distinction between procedural and conceptual knowledge is personal, context and content dependent and the results should be evaluated from this perspective. As shown by the analysis, the results vary between individuals and remind us of the fact that there exist personal dependencies. It is natural to reflect on possible factors that caused the outcome for this group of students. Relevant factors could be the students' beliefs of mathematics and their educational background. Other factors might be their learning approach and the teaching approach from their former teachers in mathematics. Factors like the use of assessments and use of calculators may also have contributed to the outcome. Whether the outcome of the study can be generalised to mathematical concepts outside functions is hard to prove. In any case, the concept of functions is central to mathematics in post-compulsory education and the conclusions are important, even if they are restricted to the concept of functions.

The outcomes of the study should also be judged in relation to possible pedagogical implications. Since there is evidence for the genetic view and the simultaneous activation view for the study group, one might say that the outcome is in favour of the developmental approach (Haapasalo & Kadijevich, 2000). The developmental approach is a reflection of the genetic view and the simulation activation view in the sense that procedural knowledge enables conceptual knowledge. Both views regard procedural knowledge as a necessary condition for conceptual knowledge. The question to ask is how we can plan instruction that enables the transition from procedural to conceptual knowledge.

The first part of this chapter is a discussion of the answers to the research questions based on the statistical analysis. Secondly, factors among the students that might have influenced the outcome are discussed also on basis of the interviews. The final discussion concerns the pedagogical implications in light of the findings.

6.1 CONCLUSIONS FROM THE STATISTICAL MODEL

This chapter comments on the main conclusions on the research questions on the basis of the statistical analysis. It is difficult, and not to be expected, to draw absolutely certain conclusions based on this analysis. However, the conclusions are supported by the collected data and provide significant results on how the different knowledge types and the ability to apply functions are related.

The research questions are:

1. How can procedural and conceptual knowledge of functions be measured?
2. How do procedural and conceptual knowledge of functions relate to each other?
3. How does the ability to apply functions relate to procedural and conceptual knowledge of functions?

The first question refers to the measurement part of the model, while the two last questions are discussed in light of the latent variable model.

6.1.1 The measurement model

It is important to be aware of the fact that we do not seek for a measure for something that exists in a positivist way. Instead we approach it by developing a test instrument based on judgments and see if the estimated model conforms to these judgments. There is of course no one true way of measuring such phenomena, but it needs to be discussed whether the selected approach seems good or bad. The relevant question here is whether the set of tasks that were developed provide a sound measure for procedural and conceptual knowledge of functions and the ability to apply functions. There seems to be a common agreement on the most typical characteristics of procedural and conceptual knowledge that should be judged when validity is considered. Validity is a complex phenomenon, but in this study the important thing is whether the questions measure what they intend to measure. In order to achieve the best possible content validity, each task was developed to meet the criteria for procedural and conceptual knowledge of functions as well as the ability to apply functions. Since content validity is impossible to measure, the data itself does not give much additional proof for content validity, except for the criteria related validity discussed in the conclusions from the post test. Since the discussion on research question number one is closely related to the task performances, some of the statistical results are given here together with the comments.

6.1.1.1 Research question 1: How can procedural and conceptual knowledge of functions be measured?

Procedural knowledge of functions is divided into two task categories, *graphic procedures* and *algebraic procedures*. One can easily agree that these are the two representation forms that are most common in a traditional mathematical context, and the students performed well with 70.1% and 71.7% correct on the two items on average. The descriptive analysis reveals that the tasks expected to be difficult had low scores and that those expected to be easier scored higher.

If we look at the questions on graphic procedures, the first question measuring those on a linear function was expected to be easy. The mean score was 3.37 of 4 (84.3%). The second question in this item was similar, but the function was a rational function that was expected to be more difficult than the first one. This was confirmed by the data as the mean score was 2.95 of 5 (59.0%), even though the difference was expected to be larger. The standard deviation was 2.88 on a scale on the range from 0 to 9, which is 32% of the range. This tells us that the tasks were suitable to detect differences between students, a property that is obviously important for a measurement tool. An estimate on Cronbach's alpha at 0.62 (Table 5-22) must also be said to be satisfactory, considering that there are only two items. Indicating a reasonable reliability, this means that it is meaningful to separate between the achievements on graphic and algebraic procedures. The scores differ enough to say that they do measure different properties, but they correlate enough to assume that they measure different aspects of the same phenomenon, in this case procedural knowledge of functions.

With only two representation forms included, all aspects are not covered in the test. One could maybe claim that the test should include tasks to measure students' procedural knowledge of functions when represented by texts or tables. However, when text or tables are used as a part of a mathematical process, it is often in a context where one translates the text into a graphic or algebraic expression. Also, it is often used to explain properties of a function rather than carrying out operations. In both cases, this has more to do with conceptual knowledge than procedural knowledge.

From Figure 5-13 one can see that the factor loading on graphic procedures is λ_1=0.52 which is lower than λ_2=0.86 loading on algebraic procedures[30]. This means that the way this measurement model is set up, algebraic procedures seems to be more strongly related to procedural knowledge of functions than graphic procedures. Of course this balance between the impact of algebraic and graphic representations is dependent on the set of tasks used in the measurement model.

Conceptual knowledge was measured with three items: relations between algebraic and graphic representations, graphic interpretations, and algebraic interpretations. As opposed to the tasks given in the previous item, these types of questions are less familiar to the students. This is easily seen in the lower scores of these items. The mean scores on the three items are 48.8%, 21.5% and 53.1% respectively.

Again, when we look at the item measuring the relations between algebraic and graphic representations, it is clear that the tasks expected to be easy have the highest percentage correct score. On task 6, where the question was related to a constant function, the mean score was 3.37 out of 4 (84.3%), while the other tasks scored around 40%. For the purpose of detecting variations, this is satisfactory.

Scores in *graphic interpretations* were very low, 4.09 out of 19 points (21.5%). This was not unexpected, as the questions were constructed in a way that did not reveal any details about procedures. Even so, the tasks did vary in degree of difficulty from 11.4% mean score on task 11 to 31.5% on task 12. Despite the low level of scores, it is possible to detect variation between the students based on the results.

The average score on algebraic interpretations was 53.1% of maximum score. The tasks are quite similar and there was a tendency that those who got one of them right, also got the other one right. This is not a weakness in the sense that they belong to the same item and are used to measure the same aspect of conceptual knowledge of functions.

Again, the different items cover different aspects. The first item has to do with transformation between representation forms, while the other two reach to some extent

[30] The loading referred to are from the model which was estimated by the correlation matrix.

the level of reification (Sfard, 1991). When the tasks were developed, covering different aspects of the same concept was intended and they do distinguish between graphic and algebraic problems. What the data tells us is that this balance seems to have been achieved to a reasonable extent. An estimated value of Cronbach's alpha equal to 0.74 (Table 5-22) suggests that the internal consistency is satisfactory.

The factor loading are λ_3=0.77 (relations between graphic and algebraic representations), λ_4=0.58 (graphic interpretations) and λ_5=0.70 (algebraic interpretations). The difference is relatively small, although graphic interpretations seem to have a slightly lower loading. Again, since these types of questions are less common, it seems reasonable that their impact is somewhat weaker.

The post test indicated that the measures for operation and conceptual knowledge of functions to some extent predicted the outcome of problems in mathematics that was supposed to rely on the two types of knowledge. This predictive ability of the measures serves as supportive evidence for the validity of the measures.

The ability to apply functions is a concept that is less commonly referred to in mathematics education. There does not seem to be a consensus on what the meaning of this concept should be, as is the case with procedural and conceptual knowledge. In summary, one can say that the ability to apply functions in the framework of this study is that one is able to apply functions in problems involving derivation and in some economics tasks. The items that measure the ability to apply functions contain problems on economic applications, derivation and graphic knowledge of derivation. All items seem to contain tasks of different difficulty. In the item that measured economic applications, task 13 had a mean score of 5.28 out of 9 (58.7%). The questions in this task are very commonly given to students in economics, and the relatively high score was expected. On the other hand, task 17 proved to cause a lot of trouble, also as one would anticipate with a mean score of only 0.64 out of 4 (16.0%). Here the problem seems to be that the students were given a lot of information and had to select which parts of the information were needed and also to combine the pieces of information to solve the problem. The item on derivation also confirmed the same tendency, with a score of as much as 3.82 out of 4 (95.5%) on task 16 *a* while 16 *d* scored 1.37 out of 4 (34.3%). Also the last item, measuring graphic knowledge of the derivative of a function, gave similar results with 7.10 out of 16 (44.4%) on task 20 and 1.38 out of 8 (17.3%) on task 21. In summary, the data confirmed my expectations regarding variation in difficulty.

The factor loadings were λ_6=0.78 (economic applications), λ_7=0.65 (derivation) and λ_8=0.62 (graphic knowledge of the derivative of a function). The impact of economic applications is marginally higher than the two others, but if we take into consideration that the two other items both concern derivation, this is only reasonable. Cronbach's alpha was estimated to be 0.72, which indicates a reasonable internal consistency between the items.

The post test indicated that the measure was a reasonable predictor for the students' achievements at their exam in business economics. This indicates that the measure of the ability to apply functions is a sound measure, at least when the ability to apply functions is restricted to applications within economics.

6.1.2 The structural model

The originally hypothesised model was adjusted, as the direct relationship between procedural knowledge of functions and the ability to apply functions seemed weak and was removed. The research question affected by the adjustment of the model is research question 3. However, none of the conclusions related to the final model contradicts the estimations of the originally hypothesised model. Given that the model is well defined and that the data meet the required conditions, the answers to research questions 2 and 3 can be illustrated by the estimated latent variable model (Figure 6-1).

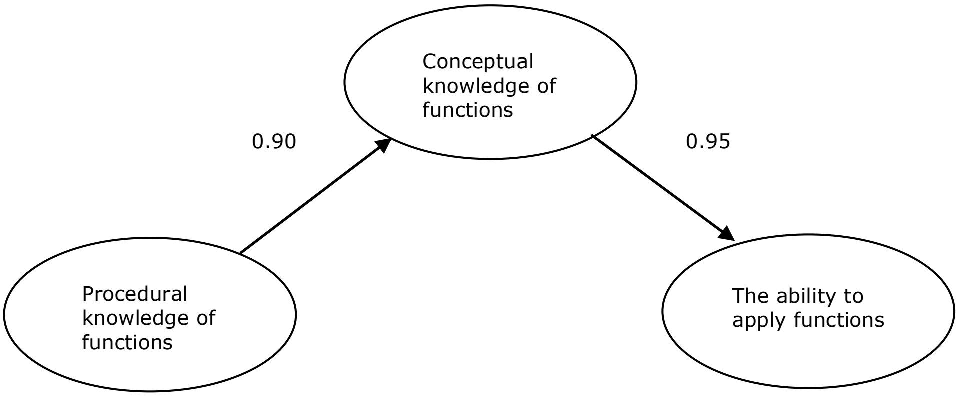


Figure 6-1. The estimated relationships between the three latent variables.

Keeping in mind that the effects are standardised similar to correlations, one can see that the effects are strong. Thus they confirm the necessity of both procedural and conceptual knowledge.

6.1.2.1 Research question 2: How do procedural and conceptual knowledge of functions relate to each other?

The relationship between conceptual knowledge of functions and procedural knowledge of functions is, as expected, very strong. It was hypothesised that procedural knowledge of functions is a necessary condition for conceptual knowledge of functions. This does not mean that conceptual development will take place simultaneously, but that this direction dominates when we talk about conceptual development for one mathematical concept in accordance with the theory of (Sfard, 2001).

In chapter 4 we formulated the hypothesis:

$H_{0;2}$: $\gamma_{11} = 0$

$H_{1;2}$: $\gamma_{11} > 0$

The regression parameter $\gamma_{11}=0.90$ ($p<0.01$) is strongly significant at any reasonable level and leaves no doubt that the relationship is strong and that $H_{0;2}$ must be rejected. Even if one might suspect that the more mathematically oriented students are more oriented towards definitions and properties than others who focus on the memorisation of procedures, this proves that students with good skills are those who have developed better conceptual knowledge.

If we accept the measurement model and the assumption that the suggested causal direction is in accordance with the genetic view (Kadijevich & Haapasalo, 2001), the conclusion is very clear on this research question. In other words, when we talk about students' conception of functions, we can assume that they need procedural knowledge to develop conceptual knowledge.

6.1.2.2 Research question 3: How does the ability to apply functions relate to procedural and conceptual knowledge of functions?

Several topics are addressed in this research question. One is whether conceptual knowledge is a necessary condition to be able to apply functions, a question which is clearly confirmed by the analysis. In chapter 4 we formulated:

$H_{0;3\text{-}2}$: $\beta_{21} = 0$
$H_{1;3\text{-}2}$: $\beta_{21} > 0$

The regression parameter $\beta_{21} = 0.95$ ($p<0.01$) is clearly significant and $H_{0;3\text{-}2}$ is rejected. Even if this part of the result is what one would expect, it is worth noting that the tendency is very strong. Again the causal direction is not proved, but the direction is a likely one. Even if examples from practice can shed light on the meaning of a concept and assist in gaining deeper mathematical understanding, it is probably natural to think that one must have conceptualized a mathematical concept to a certain extent to be able to apply it. Under any circumstances, the analysis clearly suggests that there is a very strong relationship between conceptual knowledge of functions and the ability to apply functions.

Another sub-problem in this research question is whether procedural knowledge is a necessary condition to be able to apply functions, a question that is easier to discuss if we decompose it in two questions. If we look at the direct relationship between procedural knowledge of functions and the ability to apply functions, the hypotheses are:

$H_{0;3\text{-}1}$: $\gamma_{21} = 0$
$H_{1;3\text{-}1}$: $\gamma_{21} > 0$

The analysis of the original model does not give support to conclude that this relationship is significant. The estimated regression parameter $\gamma_{21} = 0.14$ ($t=0.62$) is far from significant and $H_{0;3\text{-}1}$ cannot be rejected. As discussed in chapter 5, this relationship could be removed from the model. This does not mean that such a relationship does not exist, but in the analysis this link seems to be weak. In other words, it seems that procedural knowledge of function alone is insufficient for being able to apply functions. The other question is what was referred to as the indirect effect from procedural knowledge of functions on the ability to apply functions. In other words, procedural knowledge of functions has an indirect effect on the ability to apply functions, estimated as $\gamma_{11} \cdot \beta_{21} = 0.90 \cdot 0.95 = 0.855$ showing a clear effect. The interpretation of this is that procedural knowledge of functions has an effect on the ability to apply functions, but that is intermediated by conceptual knowledge of functions. One could say that the original model and the final model both point to the same direction, but it is a finding in itself that the direct link from procedural knowledge of functions to the ability to apply functions did not prove significant.

To summarise the analysis of the main test, procedural knowledge of functions is important, but first of all as a means to developing conceptual knowledge of functions. Conceptual knowledge of functions is in turn a condition for the ability to apply functions. The direct relationship between procedural knowledge of function and the ability to apply functions seems weak. Maybe the single most interesting part of conclusion is the importance of conceptual knowledge.

6.1.3 Conclusions from the post test

The results indicate a correspondence between performances and measures in the way that each measure, estimated as index scores, seems to be associated with the kind of performance they are supposed to predict. Each type of performance seems to have a slightly stronger association with the presumably associated index score than the others, indicating that the test discriminates the measures. It should be noted that this indication is weak and does not serve as a proof for discriminant validity. The analysis of the post test, as far as research question 1 is concerned, says that the three measures, procedural knowledge of functions, conceptual knowledge of functions and the ability to apply functions, seem to predict related performances, while the indication of discriminant validity is weaker. The post test data also support the structural relationships that were found in the analysis of the main model supporting the genetic view and the simultaneous activation view, although not as clearly as the main test indicated. The sample size of the post test was too small for a meaningful estimation of the structural equation model, but the alternative approach applied to a new dataset supports previous conclusions.

6.2 REFLECTIONS ON THE OUTCOME

It is important to notice the distinction between the conclusions from the statistical analysis and the present discussion, which is meant as a reflection on the possible factors that might have influenced the outcome. The results support the genetic view and indicate that many students have developed procedural skills, but lack conceptual knowledge. Many students seem to be procedurally-bounded (Järvelä & Haapasalo, 2005) and mainly focus on procedures. What are the possible reasons for this procedurally bounded style? Several factors may play a part, such as teacher's level of mathematical knowledge, teachers' understanding on how to teach for conceptual knowledge, students' educational background or students' beliefs. Since many students are strategically oriented and motivated by exams, the content of assessments is likely to be an important factor. Finally, some thoughts related to the role of computer environments and calculators are discussed.

6.2.1 Reflections from the interviews

The interviews provide information for the discussion of these issues based some students' description of their beliefs and educational experiences. Students' beliefs about mathematics are likely to influence their thoughts and actions as learners (Pehkonen & Safuanov, 1996). A student's belief is understood as the student's subjective knowledge and emotions about mathematics which is shaped by his or her experiences (Pehkonen & Pietilä, 2003), so it is reasonable to assume that beliefs in this regard are personal. To gain insight in students' experiences, a semi structured interview was applied to investigate students' beliefs in the nature of mathematical knowledge as well as their experiences as learners. Given the complexity of beliefs about mathematics and education, the interview is suitable to reveal thoughts not addressed in the quantitative part the study. The first part of the questions addresses beliefs about what it means to understand mathematics while the second part is related to the learning situation. The teaching approach applied at school is a factor that is likely to have influenced the student's beliefs about what it is to understand mathematics and as well as the student's approach to learning. Emma, Anna and Martin contributed with three different stories.

It is hard to see how the learning environment that Emma and Anna had experienced could be a platform for discussion and activities stimulating conceptual knowledge. The only activity they described was that of the teacher who demonstrated procedures and asked the students to replicate them on similar tasks. It seems difficult to learn and teach procedural knowledge in such environments, and almost impossible to accomplish learning activities that promote conceptual knowledge if the teacher has no clear strategy on how to promote links between procedural and conceptual knowledge.

Emma is a person who really struggles with mathematics with a purely procedural orientation. One of her remarks raises an interesting question. Her reason for focusing on remembering procedures was that she thinks mathematics is difficult. If a person with low mathematical knowledge in general may find a procedural approach as the only possible strategy, the same reasoning probably holds for teachers too. Teachers with low conceptual knowledge are unable to devise a plan to teach for deeper understanding and restrict themselves to do demonstrations of routines. This is in accordance with the findings from (Ma, 1999) that suggests that teachers in the US who intended to teach for understanding failed to do so because they did not possess deeper understanding themselves. Another aspect of Emma's story was the unsatisfactory learning environment. Too many students gathered in one room on a Friday afternoon does not encourage discussion and individual follow up.

Like Emma, Anna described herself as a student who concentrated on how to solve problems and gave similar history from upper secondary school. The teacher demonstrated solutions at the blackboard and did not follow up the pupils in class or their homework. On a couple of occasions, when given conceptual problems, she expressed that she was happy if she could do calculations. The reason for her belief could be that she found conceptual problems difficult. Another interpretation is that the conceptual task had triggered her consciousness and made a foundation for procedural action in accordance with the educational approach (Haapasalo & Kadijevich, 2000).

The initial part of the interview with Martin gives the impression that he is a conceptually oriented student in the sense that he was very motivated by relationships between mathematics on one side and other subject like physics and economics on the other. He considers himself to be a learner who focuses on rules and how problems are solved and also why they are important, as well as relational issues. However, he had more problems with the conceptually oriented exercises than procedural tasks. One possible explanation could be that his teacher from upper secondary school was mainly demonstrating how to solve problems, rather than stimulating reflections to prepare for deeper understanding. As an example, Martin told that was no synchronization between the mathematics courses and the physics courses, even with the same teacher. Another reflection is that Martin gave the impression of being a conceptually oriented learner in the beginning of the interview, but appeared to be procedurally oriented when he was working with the exercises. Putting this in the perspective of the findings of the statistical analysis it might be that students like Martin with a high level of procedural knowledge and motivation to apply mathematics might run into problems if the intermediating factor, namely conceptual knowledge is weak.

All three students reported to have procedurally oriented teachers at the upper secondary school, and despite variation, they seemed to have a better grasp on the procedural tasks. It is not surprising that they were struggling with the procedural tasks, but it might be that they are unaware of the importance of a deeper approach if they want to be able to apply mathematics. Their experiences from their teachers at the upper secondary school coincide with the report from TIMSS (2007), stating that the academic mathematical training of Norwegian mathematics teachers' at 8th grade is low. Few teachers participate in education courses relevant to their profession as teachers of mathematics. If the lack of deeper mathematical understanding among teacher hinders conceptual teaching, pupil's beliefs and learning style development is likely to be affected. To large extent, Norwegian pupils characterize the teaching of mathematics to be working individually with tasks.

This strive towards procedural strategies might be an obstacle for trying to change the focus towards relationship and properties. Schoenfeld (1982) argues that the result of believing that there is always a rule to follow may cause students not even to attempt to solve problems where they have no method. Pehkonen and Safuanov (1996) argue that pupils' beliefs work as a filter that affects their mathematical thoughts and actions. Beliefs are under constant development, influenced by experiences and other persons (Furinghetti & Pehkonen, 2002). Anna told that she had to see the relationships between economics and mathematics now that she had started to work with it. In Anna's case actions had influenced her belief and maybe changed her motivation toward working with relationships.

In summary, all three students were procedurally oriented and none of them seemed to perform better on the procedural than the conceptual tasks. Their stories from the school do not give any indication of teachers that were teaching for understanding. Despite similarities between the three stories, the interviews are a reminder that the students' knowledge profiles in terms of procedural and conceptual knowledge are individual. The interviews are not aimed to be proofs for the statistical analysis, but they did not contradict the outcome of the analysis, as the conceptual knowledge did not seem to surpass the procedural knowledge for any of the students that were interviewed.

6.2.2 Approaches to learning and influence from teachers and assessments

Approaches to learning and influence from teachers and assessments are factors that influence the learning process. These are discussed together as they are related to each other. For example, the students' approaches to learning are influenced by their teachers and their expectations to exams.

It is not surprising that many students are procedurally oriented or procedurally bounded, but a discussion on possible causes should include some reflections on their learning approach. Together with a colleague, the author conducted a test on the preferred approach to learning among students who studied a first year course in mathematics (Lauritzen & Dysvik, 2005). Altogether 47 students [31] were tested by the self-report questionnaire Approaches and Study Skills Inventory for Students (ASSIST) (Tait, Entwistle, & McKune, 1998). The questionnaire is designed to test students with respect to their learning approach. We told the students to answer the questionnaire in relation to a mathematical learning context. Two scores were calculated for each student, one on surface approach and another on deep approach. All the questions were formulated as statements, and the students were asked to which degree they agreed with the statement on a five point ordinal scale. The score 5 meant that student agreed with the statement, and 1 meant that the student disagreed. Three items "Seeking meaning", "Relating ideas" and "Use of evidence", were used to measure the deep approach to learning. Each item was given a score defined as the mean of four answers, and the concept "Surface approach to learning" was measured as the sum of scores from the three items. These were treated as interval scale variables. Similarly, the items "Lack of purpose", "Unrelated memorising" and "Syllabus-boundness" were used to measure the concept "Surface approach to learning". The two scores are plotted against each other in Figure 6-2, and show a significant negative correlation (r=-0.45, p<0.01) between the two.

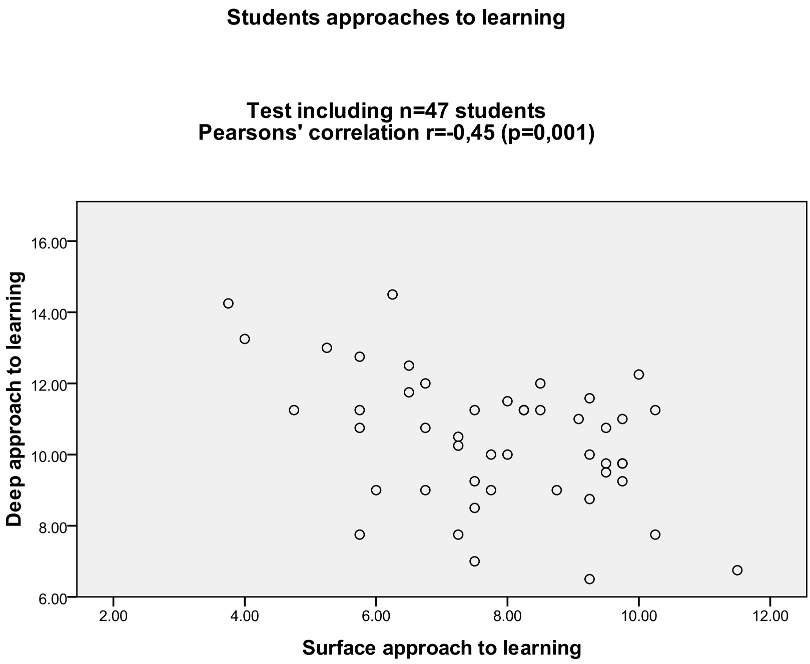


Figure 6-2. The scatter diagram indicating negative correlation between students' approaches to learning.

[31] The data collected to measure students preferred approaches to learning were taken from another group of students at Norwegian school of management BI.

The six students with the highest scores on deep approach to learning have a low score on the surface approach to learning (less than 7). Similarly, it seems like the high scorers on the surface approach to learning tend to have lower scores on the deep approach to learning. Despite the relatively small number of students (n=47), the negative correlation was significant at any reasonable level. The interpretation is that there seems to be a tendency among students to prefer one approach to the other. One should be cautious to draw conclusions on the causes for their preference of the learning approach, but a few possibilities should be commented on.

One is, as already mentioned, influenced by one's teachers. To be able to teach for understanding, the teacher must have expertise in conceptual knowledge (Ma, 1999), and to be familiar with the duality of the two knowledge types. The number of courses taken in mathematics and the level of the courses is an important factor for teachers' growth in conceptual knowledge, in particular for teachers with high procedural skills (Zerpa, Kajander, & Van Barneveld, 2009). Even if teachers at lower levels in Norway have had many years of education in general, and the picture is different for teachers at the upper secondary school, few have specialization in mathematics (TIMSS, 2007). Neither the interviews nor the test in this research addresses the level of teachers' mathematical knowledge, but a reasonable assumption is that lack of mathematical knowledge prevents teachers to benefit from their knowledge about mathematics education.

When the results were compared to performances on conceptual and procedural tasks at their final exam, the students with a deep approach to learning performed better in both procedural and conceptual tasks. The analysis in this dissertation does not test for similar relationships between learning approaches and knowledge type, but it seems obvious that in order to gain conceptual knowledge one has to look for it. An interesting reflection is that some students, like Martin who was interviewed seem to have a deep approach to learning, but have developed procedural skills more than conceptual knowledge. In this study, data was collected from a group of students that one might expect to focus on their final exam. It is obvious that many students think of what kind of problems they will meet when their learning outcome is to be evaluated. If exams measure procedural skills, no one should be surprised by the students' focus on skills in the learning situation. In that case, a strategic approach for a student could be to direct his or her attention towards memorization of procedural steps when preparing for an exam. A strategic approach refers to an intention to succeed and the motive to achieve the best grades possibly, by organizing one's time and learning environment. This kind of approach fosters an intention to achieve the best grades possible by adopting the assessment demands, driven by the motive to achieve success (Diseth, 2002). One could say that if assessments test skills, the strategic approach for a student would be to focus on memorization and rote learning. If the intention is to teach for understanding, then assessments must reflect this intention and meet the requirements for content validity. The test in this study provides examples of tasks that indicate in what way procedural knowledge can be assessed. Again, it is the conceptual learning goals that are challenging to assess, but the development of tasks that address relationships, as some of the tasks in this study, has proved successful. Such tasks can also be applied when teaching for other mathematical concepts than function. For example, Ehmke, Pesonen, and Haapasalo (2011) developed interactive online tasks used to identify students' level of procedural and conceptual knowledge of binary operations. The test consisted of tasks characterized as recognition tasks, identification tasks and production tasks that used different representations of binary operation problems.

6.2.3 Calculators and computer environments

How does the use of calculators and computers influence the understanding of mathematical concepts? This question concerns a wide area of research within mathematics education, which is not the main focus of this study. The present discussion is just meant to address a few aspects that are relevant to the analysis of this study and is not intended to cover the entire domain of computer environments.

There is no doubt that a lot of calculations in a modern society are left to computers and calculators. The calculation of prices and discounts are well known examples. The mental activities required to perform these operations manually are to some extent redundant. It seems obvious that the use of electronic equipment to perform such operations is increasing. As such, one might claim that being familiar with computer environments and calculators is a goal in itself, since it plays an important role in our everyday life. This is something quite different from applying calculators to learn mathematics. If we think of a calculator as a pedagogical tool, other aspects should be considered. The curriculum for upper secondary education ("Curriculum for Upper Secondary Education; Specialized Subjects in General and Business Studies: Mathematics", 2000), says:

> *"It is difficult to say to what extent further developments will influence the need for calculation skills, and there is much debate about this, but there is no doubt that some basic skills will always be needed to formulate and adapt mathematical problems for computer processing."*

The last part of the sentence from the curriculum relates to the first two steps in Polya's (1945) model on problem solving, understanding the problem and conceiving a plan for its solution. It is the third step, carry out the solution, which is often performed by computer processing. The fourth step, reflecting on the solution, cannot obviously be left to a computer or a calculator alone. If calculations are left to a calculator, is it possible that this will prevent students from developing sufficient procedural knowledge? The rationale for this question is based on the assumption that working through the procedures manually is often no longer needed. Many of the tasks used in this study to measure procedural knowledge, can be solved by using a calculator. One example is Task 2(2) seeking to measure *graphic procedures* (x_1):

Task 2

The function g(x) is given by $g(x) = x + \frac{1}{x}$

Sketch the graph of g(x)

Figure 6-3. Task on graphic procedures.

The student will have to enter the algebraic expression, and the calculator can easily draw the graph. In this case, the calculation of function values will take a fraction of a second and the student will not have to perform the calculations manually. The details in algorithms are taken care of by the calculator. Tall (1994) talks about external, analogue and specific insight to the algorithms involved. External insight occurs when the user knows how to use the calculator, but is unknown to how the algorithm works. Specific insight, on the other, hand is when the user is fully aware of the algorithms. Analogue insight is something in-between, meaning that the user has some idea of the algorithms. One should of course be careful to conclude that the students are less trained in performing algorithms, but one should be aware that at least some of the procedures only require that the student have external insight in how the calculator works.

Broman (1996) claims that over 90% of mathematics time is commonly used on step 3 in Polya's model (carrying out the solution), a stage that is associated with procedural knowledge. Even if it is hard to quantify such time-consumption exactly, it seems obvious that for example sketching the graph in task 2 takes a lot more time when performed manually. This suggests that time is saved when a calculator is applied, and therefore more time can be spent on the other parts of the problem solving process.

Even if calculators and computer environments can produce an answer in algorithmic way when we feed them with the correct input, it is still too easy to say that their relevance in the process of learning mathematics is limited to performance of procedures. How can use of calculators and computer environments influence on development of knowledge? Maybe a more important question is how can computers establish links between procedural and conceptual knowledge? The MODEM environment implemented by Haapasalo (Kadijevich & Haapasalo, 2001) is an example of a computer environment that enables procedural knowledge development by utilizing conceptual knowledge through representational transformations. The MODEM environment is based on the educational approach assuming that conceptual knowledge enables procedural knowledge. Since this study supports the view that procedural knowledge of functions is a necessary condition for conceptual knowledge of functions, development of conceptual knowledge will presumably suffer if calculators prohibit the development of procedural knowledge. However, the picture is probably more complex than this. Cates (2002) conducted a study to investigate whether students conceptual knowledge of functions were influenced by the use of a computer based laboratory. The experimental group (n=29) that took part in computer-based activities was compared to a control group (n=27) that did not take part in such activities. Cates found out that the laboratory group had significant better achievements in modelling real world phenomena with functions, interpreting functions, translating between different representational forms and in reifying functions. The two first results refer to the ability to apply functions, while the last two address the conceptual knowledge of functions. This is an interesting result, since it can be interpreted to mean that conceptual knowledge and the ability to apply a concept benefit from the use of technology, but it does not conclude how it influenced procedural knowledge.

Reflections about relationships are criteria of conceptual knowledge where computers or calculators might have an impact. Graphic calculators allow students to explore the relationship between an algebraic expression of a function and its graph. Tall and Winkelmann (1988) say that this external insight provides knowledge to check whether the results are sensible. In the example in task 2, the graph would approach the vertical line $x=0$ asymptotically, which corresponds with the fact that $g(x)$ is undefined for $x=0$. In other situations, calculators are less suited as tools to think about results. If we look at the tasks that were used to measure conceptual knowledge of functions, the process of solving some of them would not benefit from the use of a graphic calculator. Task 10 was used to measure *graphic interpretations* (y_2):

Task 10

The graph of f(x) is shown below. Sketch the graph of f(-x). You don't need to put more numbers on the axis. A rough sketch is enough.

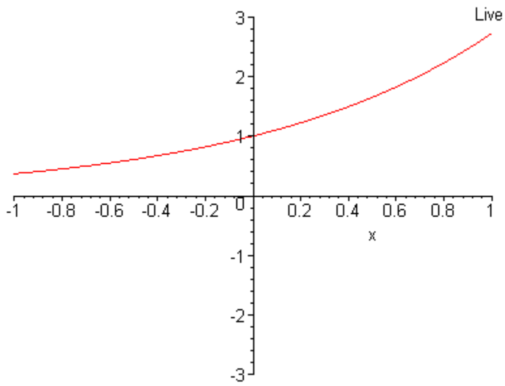


Figure 6-4. Task on graphic interpretations.

Since the students were not given an algebraic expression they would have problems to produce the graph on the calculators display. However, this example does not justify the claim that calculators do not play a part in development of conceptual knowledge. Rather it illustrates that in some situations where conceptual knowledge is required, calculators will not help to solve the problem.

Some of the tasks used to measure the ability to apply functions are not as purely procedural or conceptual, as is the case in tasks 2 and 13. Again, referring to Polya's (1945) stages in problem solving. I will use task 13 as an example on how calculator might play a role. Task 13 was used to measure *economic applications* (y_4):

Task 13

The cost of producing x units of a product is given by

$K(x) = -0{,}1x^2+6x+200$ when x is in the interval [0,20].

Calculate the marginal cost K'(x).
Estimate the marginal cost for x=10. What is the interpretation of this number?
Estimate the marginal cost for x=15. What is the interpretation of this number in relation to the answer you got in the previous question?

Figure 6-5. Task on economic applications.

Understanding the problem might be easier if one started to draw the graph of K(x) on the calculator. If the syntax is correct, the calculator will show the correct graph. Many students tend to draw graphs incorrectly, even for simple functions, when they do it by hand. One of the characteristics for conceptual knowledge is the ability to construct links between different representation forms, and this study clearly indicates that the students have problems with the interpretation of graphs. It might be that some students interpret the graph as something external to the function (Vinner & Dreyfus, 1989), and they do not realize how the graph expresses relations between the variables. Assuming that a student has overcome these problems, the graph provided by the calculator is easier to produce and more reliable than the one produced by hand. Conceiving a plan for the solution is about selecting the appropriate algorithm, given that the problem is fairly well understood. It is more difficult to see how a calculator could contribute to the solution in this case. Since the student is explicitly asked to calculate the marginal cost K'(x), the problem of selecting a procedure is practically absent. A calculator cannot compute K'(x) unless it is symbolic, but it can calculate the requested marginal costs effectively. When it comes to reflection on the result the original graph of K(x) may be beneficial, even if reading the slopes from K(x) may be difficult. Altogether, the calculator may have an impact on the different stages of solving problems like the one in task 13, primarily because it provides accurate and reliable graphic representations.

Even relationships between tabulated variables can be represented graphically. In this way, calculators can be used to experience relationships between representation forms. It is therefore reasonable to assume that computers and calculators might have an impact on the cognitive processes that lead to both procedural and conceptual knowledge of mathematical concepts.

The development from procedural to conceptual knowledge where procedures are routinized and encapsulated describes the development of conceptual knowledge as what Piaget (Tall, 1994) refers to as vertical growth. Vertical growth is different from horizontal growth, in which conceptual development takes place by focus on different representations. Computers may, according to Tall, play an important role in horizontal development in the sense that calculators or computers allow the student to reflect on the results immediately. In this way they allow the learner to reflect on different representations simultaneously. The point is that the students will be able to see how a change in one of the representations immediately causes changes to another. Tall underscores that mental objects learned this way may have different structures in comparison to others learned in a more traditional way. Teaching and learning mathematics horizontally by the use of computers will require that the students know how the program is used, more than specific knowledge on the built in procedures. In this study, I have argued for a cause of direction from procedural to conceptual knowledge of functions, but the claim that detailed knowledge about the internal routines in the calculator is redundant for horizontal development to take place, clearly challenges this view. On the other hand, uncertainty about this causal direction will not affect the main conclusions in this study.

6.3 QUESTIONS ADDRESSED BY THE STUDY

To improve teaching, we have to study how students learn, and apply this knowledge in our teaching. It is not a trivial task to change ones teaching strategy even if new knowledge on students' conceptual development is attained. Assuming that both procedural and conceptual knowledge are important for the student, the challenge for many teachers is probably related to the second of these. It seems reasonable to claim that traditional teaching emphasizes the mastery of skills, perhaps at the cost of conceptual development. A teaching approach aiming directly at conceptual knowledge by giving definitions and talking about properties without focus on procedures would contradict the view that procedural knowledge is a requirement for conceptual knowledge. On the other hand, focusing entirely on procedures without drawing the attention towards relational issues or properties is unlikely to be a successful strategy to teach for conceptual knowledge, as the results from this study indicate in the case of functions. It is difficult to conceive a plan for education without having a picture of the state of students' knowledge and an idea of what brought them there.

Students in economics are mainly interested in learning economics and many of its phenomena are explained mathematically. This research proves that a certain level of conceptual knowledge is required to be able to apply mathematics and that procedural knowledge alone is insufficient. Many seem to have a lack of conceptual knowledge and hence they will struggle to achieve the goal of understanding economics, the intermediating factor is the problem. Another conclusion is that the students in the study can be categorised with respect to procedural-conceptual links by comparing their performances on performances on procedural and conceptual tasks. This research shows that students have different profiles related to procedural and conceptual knowledge. Almost none of the students in this study with low procedural knowledge scored high on conceptual knowledge. However, many students scored high on procedural knowledge, but low on conceptual knowledge.

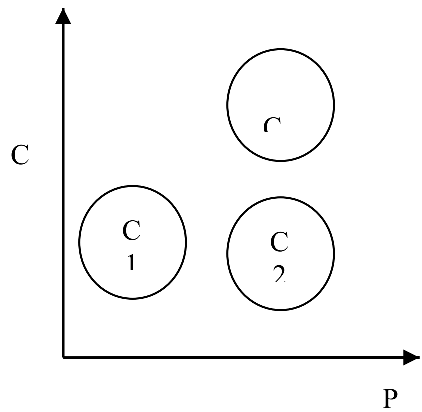


Figure 6-6. Categories of procedural (P) and conceptual (C) knowledge among students.

The students in this study can be divided in three categories based on their performances as illustrated in Figure 6-6. Many students seem to end up in categories C1 and C2. This raises two questions: what caused the cases C1 and C2 and how could we plan for education to promote development from C1 and C2 to C3? Possible causes related to the first question are discussed by mentioning some factors. The present study does not provide any evidence of the level of consciousness among teachers about what procedural and conceptual knowledge really is. Neither does it say anything about the teachers' mathematical knowledge. Nevertheless, it can be beneficial to characterise students' educational background to better understand how pedagogical approaches can be improved. When Järvelä and Haapasalo (2005) classify three types of learners, the *conceptually oriented learner*, the *procedurally oriented* learner and the *procedurally bounded learner*, the idea is that instruction can be tailored to meet the needs of different learners.

The conceptually oriented learner tries to learn principles first, then procedures, while the procedurally oriented learner seems to use procedures before bringing in principles. The last category, the procedurally bounded learner is focused on procedures without development towards conceptual learning. The data in this study clearly suggest that many of the students are procedurally oriented or even procedurally bounded.

How can we plan a new kind of pedagogical approach that promotes conceptual learning for those in category C1? One alternative could be to emphasize transformations between different representations. The study (Pesonen, Haapasalo, & Lehtola, 2002) presents software that teachers can utilize to connect algebraic and graphic representation to promote conceptual learning of functions. The second question is how we could promote learning activities for students in C2 to benefit from the procedural knowledge that they already gained. If conceptual development emerges through a back and forth process between procedural and conceptual activities, then the learning activities should be planned so that one could benefit from such an iterative process.

The statistical analysis looks at procedural and conceptual knowledge of functions at an aggregate level for a group of students. Since there is substantial variation between individuals, teaching strategies should ideally be targeted towards individuals. With large groups of students it is not realistic to develop educational programs at an individual level, while differentiation at group level is more feasible. Different learning activities could be administered for students in the different categories as indicated in Figure 6-6. This differentiation idea depends on a diagnostic tool to measure students' procedural and conceptual knowledge. A tool based on a confirmatory factor analysis, like the one in this dissertation, has shown to be a possible approach to develop measurement instruments for this purpose.

6.4 FINAL REMARKS

In this study, a statistical model with the intention of measuring the concepts that make up procedural knowledge of functions, conceptual knowledge of functions and the ability to apply functions was developed. A structural modeling technique was used to develop a measurement instrument that seems to be valid and reliable. I claim that structural equation modelling is an appropriate way to study mathematical concept building, as the concepts we are studying are often vague and difficult to analyse by more traditional modelling techniques. The path diagram serves as a communicative tool for talking about aspects of knowledge. This study shows that it is possible to develop valid and reliable tasks to measure procedural and conceptual knowledge of a mathematical concept.

Realizing that conceptual knowledge is a main goal of mathematics education should not allow us to overlook the role of procedural skills to achieve this goal. Instead of saying that we should not focus on "how" but "why", it would maybe be better to say that we should not *only* focus on "how", but on "how and why". The discussion on possible impact of factors that are not embedded in the model, such as teaching practices and the use of assessments and calculators, may provide a foundation for the hypotheses of further studies.

This study was restricted to the knowledge of functions and to some economic applications. It is tempting to transfer the conclusions to other mathematical concepts and other domains of applications, but such connections are not proven by the present study. The data in the present study was collected from students at a business school and the conclusions from the analysis must therefore be interpreted mainly with this population in mind.

The majority of research within mathematics education is based on studying children rather than adult students. A lot of conditions may differ between children and students in their twenties. Students take lectures in large classes, making it difficult for the teacher to interact with them on an individual level during a class. The risk for misconceptions to pass without being corrected is probably greater. To what extent can theories regarding conceptual development be transferred to college and university students? Older students' enhanced capacity to reflect on their own activities might make them more flexible in their approach to a mathematical problem. It might be that social factors, learning strategies, learning environment, misconceptions, assessments, cognitive aspects, anxiety and other things are factors that may have a different impact on students than among younger children. Robert and Schwarzenberger (1991) conclude that it is hard to find features that are specific to students, but that it rather is a question of quantity. Piaget (1977) looked at the development of mathematical understanding through what he called reflective abstraction. Even if Piaget studied younger children's construction of knowledge, Dubinsky (1991) claims that the same ideas can be extended to a general theory which is applicable for students at a higher level.

Understanding a mathematical concept involves the ability to see relations for example between graphic and algebraic representation of a mathematical concept. Maybe the same is true for understanding conceptualization, namely to see the relations between the types of understanding in addition to understand them separately. The duality between procedural and conceptual knowledge is approached from two perspectives in this dissertation. The problem is discussed at a conceptual level by looking at relations to previous research, at the nature and properties of knowledge types. In the final conclusions, relationships to other aspect such as students' beliefs are discussed. The other perspective is a statistical analysis of a large sample that can be associated with a procedural approach. If we do understand the relation between those approaches, and not only focus on one of them, we might understand the duality in mathematical knowledge better.

References

Ary, D., Jacobs, L. C., & Razavieh, A. (1996). *Introduction to Research in Education* (5 ed.): Harcourt Brace College.

Bagozzi, R. P., & Yi, Y. (1988). On the Evaluation of Structural Equation Models. *Jornal of the Academy of Marketing Science, 16*(1), 74-94.

Baker, W., & Czarnocha, B. (2002). *Written Metacognition and Procedural Knowledge.* Paper presented at the Proceedings of the 2nd International Conference on the Teaching of Mathematics, University of Crete, Hersonissos.

Bentler, P. M., & Bonett, D. G. (1980). Significance tests and goodness of fit in the analysis of covariance structures. *Psychological Bulletin, 88*, 588-606.

Berger, P. L., & Luckmann, T. (1996). *The Social Construction of Reality*. New York: Anchor.

Biggs, J. (1993). What do inventories of students' learning processes really measure? A theoretical review and clarification. *British Journal of Educational Psychology, 63*(3-19).

Blalock, H. M. (1982). *Conceptualization and Measurement in the Social Sciences*. Beverly Hills: Sage.

Bollen, K. A. (1989). *Structural equations with latent variables* New York: Wiley.

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics, 23*, 247-285.

Broman, P. (1996). Possibilities and Fears. In P. Gómez & B. Waits (Eds.), *Roles of Calculators in the Classroom*, 15-20.

Browne, M. W., & Cudeck, R. (1993). Alternative ways of assessing model fit. In K. A. Bollen, & Long, J. S. (Ed.), *Testing Structural Equation Models*. Newbury Park, CA: Sage, 136-162.

Byrne, B. M. (1998). *Structural equation modeling with LISREL,PRELIS, and SIMPLIS :Basic Concepts, Applications, and Programming* London: Erlbaum.

Byrnes, J. P., & Wasik, B. A. (1991). Role of Conceptual knowledge in Mathematical Procedural Learnig. *Developmental Psychology, 27*(5), 777-786.

Carpenter, T. P. (1986). Conceptual Knowledge as a Foundation for Procedural Knowledge. In J. Hiebert (Ed.), *Conceptual and Procedural Knowledge: The Case of Mathematics*. Hillsdale, NJ: Erlbaum, 113-132.

Cates, B. B. (2002). *CBL Activities and Students' Conceptual Knowledge of Functions*. Paper presented at the International Conference on Technology in Collegiate Mathematics.

Chiang, A. C., & Wainwright, K. (2005). *Fundamental Methods of Mathematical Economics* (4 ed.). New York: McGraw-Hill.

Cobb, P. (1988). The Tension Between Theories of Learning and Instruction in Mathematics Education. *Educational Psychologist, 23*(2), 87-103.

Cramer, D. (2003). *Advanced Quantitative Data Analysis*. London: Open University Press.

Crawford, K., Gordon, S., Nicholas, J., & Prosser, M. (1994). Conceptions of mathematics and how it is learned; the perspectives of students entering university. *Learning and Instruction, 4*, 331-345.

Cronbach, L. J. (1951). Coefficient Alpha and the Internal Structure of Tests. *Psychometrica, 16*, 297-334.

Curran, P. J., West, S. G., & Finch, J. F. (1996). The robustness of test statistics to nonnormality and specification error in confirmatory factor analysis. *Psychological Methods, 1*, 16-29.

Curriculum for Upper Secondary Education; Specialized Subjects in General and Business Studies: Mathematics (2000). Oslo, February 2000. Ministry of Education.

Davis, R. B. (1992). Understanding "Understanding". *Journal of Mathematical Behavior, 11*, 225-241.

Diseth, Å. (2002). *Approaches to learning: Validity and prediction of academic performance. Unpublished Doctoral thesis*. Bergen: University of Bergen.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. Tall (Ed.), *Advanced Mathematical Thinking*. Dordrecht: Kluwer, 95-126.

Dubinsky, E., & Harel, G. (1992). The Nature of the Process Conception of Function. In G. Harel, & Dubinsky, E. (Ed.), *The concept of function: Aspects of epistemology and pedagogy* (Vol. 25). Washington: Mathematical Association of America.

Duffy, T. M., & Jonassen, D. H. (1992). *Constructivism and the technology of instruction: A conversation*. Hillsdale, NJ: Erlbaum.

Dörfler, W. (1999). Means for Meaning. In P. Cobb, Yackel, E., & McClain, K. (Ed.), *Symbolizing and Communicating in Mathematics Classrooms: Perspectives on Discourse, Tools and Instructional Design*. Mahwah, NJ: Erlbaum, 99-132.

Edwards, L. D. (1998). Embodying Mathematics and Science: Microworlds as Representations *Journal of Mathematical Behavior, 17*(1), 53-78.

Ehmke, T., Pesonen, M., & Haapasalo, L. (2011). Assessment of university students' understanding of abstract binary operations. *Nordic Studies in Mathematics Education, 15*(4), 25-40.

Entwistle, N., & Tait, H. (1990). Approaches to learning, evaluations of teaching, and preferences for contrasting academic environments. *Higher Education, 19*, 169-194.

Eronen, L., & Haapasalo, L. (2010). Making Mathematics through Progressive Technology. In: Sriraman Bharath, Bergsten Christer, Goodchild Simon, Palsdottir Gudbjörn, Dahl Bettina (Eds.), *The First Sourcebook on Nordic Research in Mathematics Education*. Charlotte, NC: Information Age Publishing, 701-710.

Fischbein, E. (1993). The interaction between the formal, the algorithmic and the intuitive components in a mathematical activity. In R. Biehler, Scholz, R. W., Strasser, R., Winkelmann, B. (Ed.), *Didactics of mathematics as a scientific discipline*. Dordrecht: Kluwer, 231-245.

Fornell, C., & Larcker, D. F. (1981). Evaluating Structural Equation Models with Unobservable Variables and Measurement Error. *Journal of Marketing Research, XVIII*, 39-50.

Furinghetti, F., & Pehkonen, E. (2002). Rethinking Characterizations of Beliefs. In G. Leder, E. Pehkonen & G. Törner (Eds.), *Beliefs: A Hidden Variable in Mathematics Education?* Dordrecht: Kluwer, 39-57.

Geffen, D. (2003). Assessing Unidimensionality through LISREL: An Explanation and Example. *Communications of the Association for Information Systems, 12*, 23-47.

Gray, E., & Tall, D. (1993). Success and Failure in Mathematics: The Flexible Meaning of Symbols as Process and Concept. *Mathematics Teaching 142*, 6-10.

Gray, E., & Tall, D. (1994). Duality, Ambiguity, and Flexibility: A "Proceptual" View of Simple Arithmetic. *Journal of Research in Mathematics Education, 25*(2), 116-140.

Hair, J. F., Anderson, R. E., Tatham, R. L., & Black, W. C. (1998). *Multivariate Data Analysis*. NJ: Prentice Hall.

Herscovics, N., & Bergeron, J. C. (1983). Models of understanding. *International Reviews on Mathematics Education, 15*(2), 75-83.

Hiebert, J. (1986). *Conceptual and procedural knowledge: the case of mathematics*. Hillsdale, NJ: Erlbaum.

Hiebert, J., & Lefevre, P. (1986). Conceptual and Procedural Knowledge in Mathematics: An Introductory Analysis. In J. Hiebert (Ed.), *Conceptual and Procedural Knowledge: The Case of Mathematics*. Hillsdale, NJ: Erlbaum, 1-27.

Hiebert, J., & Wearne, D. (1986). Procedures Over Concepts: The Acquisition of Decimal Number Knowledge In J. Hiebert (Ed.), *Conceptual and Procedural Knowledge: The Case of Mathematics*. Hillsdale, NJ: Erlbaum, 199-223.

Haapasalo, L. (1993). Systematic Constructivism in Mathematical Concept Building. In P. Kupari & L. Haapasalo (Eds.), *Constructivist and Curricular Issues in the School Mathematics Education. Mathematics Education Research in Finland. Yearbook 1992-1993*. University of Jyväskylä. Institute for Educational Research. Publication Series B, Theory and Practice 82. University of Jyväskylä & The Finnish Association of Mathematics and Science Education Research, 9-22.

Haapasalo, L., & Kadijevich, D. (2000). Two Types of Mathematical Knowledge and Their Relation. *Journal für Mathematik-Didaktik 21*(2), 139-157.

Haapasalo, L., & Samuels, P. (2011). Responding to the Challenges of Instrumental Orchestration through Physical and Virtual Robotics. *Computers & Education, 57*(2), 1484-1492.

Jackson, P. W. (1996). *The Practice of Teaching*. New York: Teachers College Press.

Janvier, C. (1978). *The interpretation of complex cartesian graphs representing situations-studies and teaching experiments.* University of Nottingham, England.

Järvelä, J., & Haapasalo, L. (2005). *Three Types of Orientations by Learning Basic Routines in ICT.* Paper presented at the Learning and Instruction on Multiple Context and Settings III. Proceedings of the Fifth Joensuu Symposium on Learning and Instruction. , Joensuu.

Jöreskog, K., & Sörbom, D. (1993). *LISREL 8: Structural Equation Modeling with the SIMPLIS Command Language*. Hillsdale, NJ: Erlbaum.

Jöreskog, K. G. (1973). A general method for estimating a linear structural equation system. In A. S. Goldberger, & Duncan, O. D. (Ed.), *Structural Equation models in the Social Sciences*. New York: Academic Press, 85-112.

Jöreskog, K. G. (1977). Structural equation models in the social sciences: specification, estimation and testing. In P. R. Krishnaiah (Ed.), *Applications of Statistics*. Amsterdam, 265-287.

Jöreskog, K. G., & Sörbom, D. (1988). *PRELIS - A program for multuvariate data screening and data summarization. A preprocessor for LISREL* (2 ed.). Chicago: Scientific Software International.

Jöreskog, K. G., & Sörbom, D. (1996). *PRELIS 2: User's Reference Guide* Chicago: Scientific Software International.

Kadijevic, D. (1999). Conceptual Tasks in Mathematics Education. *The Teaching of Mathematics, 2*(1), 59-64.

Kadijevich, D., & Haapasalo, L. (2001). Linking procedural and conceptual mathematical knowledge through CAL. *Journal of Computer Assisted Learning, 17*(2), 156-165.

Kaput, J. (1982). *Differential effects of the symbol systems of arithmetic and geometry on the interpretation of algebraic symbols*. Paper presented at the Annual Meeting of the American Educational Research Association, NY.

Kickbusch, K. (1996). *Teaching for Understanding: Educating Students for Performance.* Retrieved in October 2009 from Internet: http://www.weac.org/News_and_Publications/education_news/1996-1997/under.aspx.

Kitcher, P. (1983). *The nature of mathematical knowledge*. NT.

Kline, M. (1980). *Mathematics: The Loss of Certainty*. New York: Oxford University Press.

Lauritzen, P., & Dysvik, A. (2005). *Relations between approaches to learning and conceptual understanding in mathematics.* Paper presented at the Fourth Nordic Conference on Mathematics Education, Trondheim.

Ma, L. (1999). *Knowing and Teaching Elementary Mathematics Teachers' Understanding of Fundamental Mathematics in China and the United States*. Mahwah, NJ: Erlbaum.

Marton, F., & Säljö, R. (1976). On qualitative differences in learning: I. Outcome and process. *British Journal of Educational Psychology, 46*(1), 4-11.

McDonald, R. P. (1989). An Index of Goodness of Fit Based on Noncentrality. *Journal of Classification, 6*, 97-103.

Mellin-Olsen, S. (1981). Instrumentalism as an Educational Concept. *Educational Studies in Mathematics, 12*, 351-367.

Ministry of Education, R. a. C. A. (2000). Curriculum for Upper Secondary Education; Specialized Subjects in General and Business Studies: Mathematics In R. a. C. A. Ministry of Education (Ed.). Norway.

Maass, J., & Schlöglmann, W. (2009). Beliefs and Attitudes in Mathematics Education. Retrieved in August 2012 from https://www.sensepublishers.com/files/9789087907235PR.pdf.

Nantais, N., Herscovics, N., & Bergeron, J. C. (1984). The skills-understanding dilemma in mathematics education. In J. M. Moser (Ed.), *Proceedings of the sixth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Madison: Wisconsin Center for Education Research, 229-235.

Nesher, P. (1986). Are Mathematical Understanding and Algorithmic Performance Related. *For the Learning of Mathematics, 6*(3), 2-9.

Newble, D., & Clarke, R. (1987). Approaches to learning in a traditional and an innovative medical school. In J. T. E. Richardson (Ed.), *Student Learning: Research in Education and Cognitive Psychology*. Milton Keynes: Open University Press, 39-48.

Niss, M. (1998). Aspects of the Nature and State of Research in Mathematics Education. *Documenta Mathematica, Extra Volume ICM, Vol III*, 767-776.

Noss, R. (1998). New numeracies for a technological culture. *For the Learning of Mathematics, 18*(2), 2-12.

Olsson, U. H., Troye, S. V., & Howell, R. D. (1999). Theoretical fit and empirical fit: The performance of maximum likelihood versus generalized least squares estimation in structural equation modeling. *Multivariate Behavioral Research, 34*(1), 31-59.

Pajares, F., & Miller, D. (1994). Role of Self-Efficacy and Self-Concept Beliefs in Mathematical Problem Solving: A Path Analysis. *Journal of Educational Psychology, 86*, 193-203.

Pehkonen, E., & Pietilä, A. (2003). *On Relationships between Beliefs and Knowledge in Mathematics Education.* Paper presented at the CERME-3, Bellaria (Italy).

Pesonen, M., Haapasalo, L., & Lehtola, H. (2002). Looking at Function Concept through Interactive Animations. *The Teaching of Mathematics, 5*(1), 37-45.
Piaget, J. (1977). *The development of thought : equilibration of cognitive structures* New York: Viking Press.
Polya, G. (1945). *How to Solve It*. Princeton: Princeton University Press.
Preacher, K. J., & Leonardelli, G. J. (2006). Calculation for the Sobel Test: An interactive calculation tool for mediation tests, from http://people.ku.edu/~preacher/sobel/sobel.htm
Progress, N. A. O. E. (1983). The third national mathematics assessment: Results, trends, and issues. Denver, CO: Educational Commission of the States.
Resnick, L., & Omanson, S. F. (1987). Learning to Understand Arithmetic. In G. R. (Ed.), *Advances in Instructional Psychology* (Vol. 3). Hillsdale, NJ: Erlbaum.
Robert, A., & Schwarzenberger. (1991). Research in Teaching and Learning Mathematics at an Advanced Level. In D. Tall (Ed.), *Advanced Mathematical Thinking*: Dordrecht: Kluwer, 127-139.
Scheffler, I. (1965). *Conditions of knowledge: An introduction to epistemology and education*. Chicago: University of Chicago Press.
Schoenfeld, A. H. (1982). Learning to Think Mathematically: Problem Solving, Metacognition, and Sense Making in Mathematics. In D. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning: a Project of the National Council of Teachers of Mathematics*. New York: Macmillan, 334-370.
Schoenfeld, A. H. (1990). *A source book for college mathematics teaching*. Washington DC: Mathematical Association of America.
Sfard, A. (1991). On the Dual Nature of Mathematical Conceptions: Theoretical Reflections on Processes and Objects as Different Sides of the Same Coin. *Educational Studies in Mathematics, 22*, 1-36.
Sfard, A. (1994). Reification as a birth of a metaphor. *For the Learning of Mathematics, 14*(1), 44-55.
Sfard, A. (2001). There is More to Discourse than Meets the Ears: Learning from mathematical communication things that we have not known before. *Educational Studies in Mathematics, 46*, 13-57.
Skemp, R. R. (1976). Relational Understanding and Instrumental Understanding *Mathematics Teaching, 77*, 20-26.
Sobel, M. E. (1982). Asymptotic intervals for indirect effects in structural equations models. In S. Leinhart (Ed.), *Sociological methodology*. Washington, DC: American Sociological Association, 290-312.
Spearman, C. (1904). General intelligence, objectively determined and measured. *American Journalof Psychology, 15*, 201-293.
Steiger, J. H. (1990). Structural model evaluation and modification: An interval estimation approach. *Multivariate Behavioural Research, 25*, 173-180.
Steiger, J. H., Lind, J.C. (1980). *Statistically based tests for the number of common factors*. Paper presented at the The Psychometric Society Annual Meeting, IOWA City, IA.
Steiger, J. H., & Schönemann, P. H. (1978). A history of factor indeterminancy. In S. S. (Ed.), *Theory construction and data analysis in the behavioral sciences*. San Francisco: Jossey-Bass.
Studenmund, A. H. (2001). *Using Econometrics: A Practical Guide* (4 ed.). New York: Addison Wesley.

Tait, H., Entwistle, N., & McKune, V. (1998). ASSIST: A reconceptualisation of the Approaches to Studying Inventory. In R. C. (Ed.), *Improving students as learners*. Oxford: Oxford Centre for Staff and Learning Development.

Tall, D. (1991). The Psychology of Advanced Mathematical Thinking. In D. Tall (Ed.), *Advanced Mathematical Thinking*. Dordrecht: Kluwer, 3-21.

Tall, D. (1994). Computer Environments for the Learning of Mathematics. In R. W. S. R.Bieler, R. Strässer, B. Winkelmann (Ed.), *Didactics of Mathematics as a Scientific Discipline*. Dordrecht: Kluwer, 189-199.

Tall, D., & Winkelmann, B. (1988). Hidden algorithms in the drawing of discontinous functions. *Bulletin of the I.M.A., 24*, 111-115.

Thurnstone, L. L. (1947). *Multiple-Factor Analysis*. Chicago: University of Chicago Press.

TIMSS, T. i. I. M. a. S. S. (2007). Tegn til bedring?

Vergnaud, G. (1990). Epistemology and Psychology of Mathematics Education. In P. K. Nesher, J. (Ed.), *Mathematics and Cognition: A Research Synthesis by the International Group for the Psychology of Mathematics Education*. Cambridge: Cambridge University Press, 14-30.

Vinner, S., & Dreyfus, T. (1989). Images and Definitions for the Concept of Funtions. *Journal for Research in Mathematics Education, 20*(4), 356-366.

Wright, S. (1934). The method of path coefficients. *Annals of Mathematical Statistics, 5*, 161-215.

Zerpa, C., Kajander, A., & Van Barneveld, C. (2009). Factors that Impact Preservice Teachers' Growth in Conceptual Mathematical Knowledge During a Mathematics Methods Course. *International Electronic Journal of Mathematics Education, 4*(2), 57-76.

Zucker, B. (1984). *The relation between understanding and algorithmic knowledge in decimals.* (M. A. Thesis), University of Haifa.

Appendix A

THE TEST GIVEN IN THE STUDY

Task 1. Sketch the graph of h(x) = 2x – 1

Task 2. The function g(x) is given by g(x) = $x + \frac{1}{x}$

Calculate the value for g(x) when x = -5, x= -2 , x = -1, x = 1, x = 2 and x = 5
Sketch the graph of g(x)

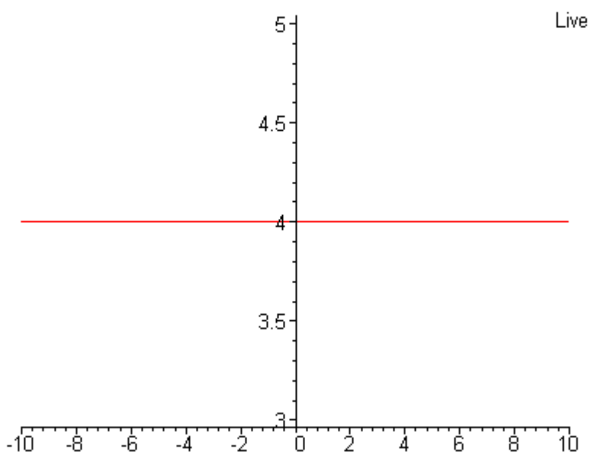
Task 3. Given f(x) = - x – 3. For which value of x is f(x) = 0 ?

Task 4. In this task we look at the function $f(x) = 2x^2 - 8x + 6$, Df = R

Calculate f(x) when x = -1 and when x = 4
When is f(x) = 0 ?
When is f(x) $\leq$ 0 ?

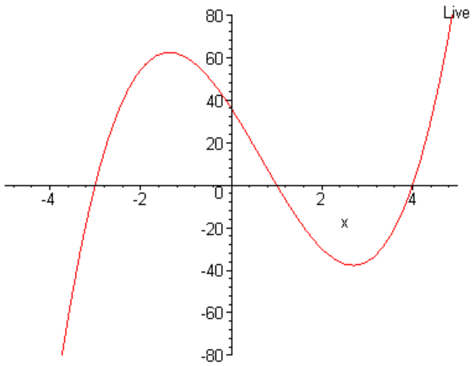
Task 5. The cost for a company to employ a person is the salary in addition to other costs (taxes) estimated to be 40% of the employee's salary. A company's total cost for an employee is 500.000. What is the salary?

Task 6. The graph of f(x) is shown below. Write down the expression for f(x).



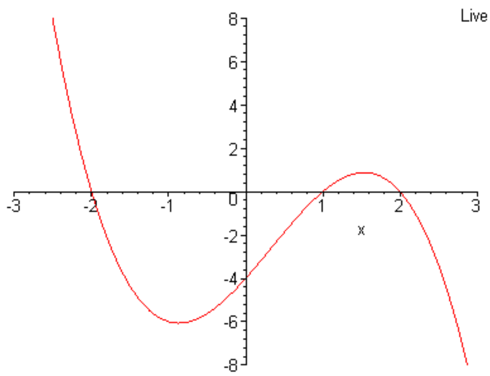
Task 7. The graph of a function is shown below. Which of the following expression can the function be divided by:

(x+1) (x+2) (x+3) (x+4) (x+5) (x-1) (x-2) (x-3) (x-4) (x-5) ?

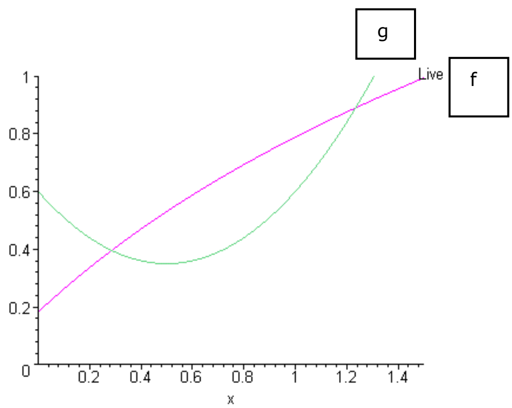


Task 8. A function of third degree has the form $f(x)=ax^3 + bx^2 + cx + d$.

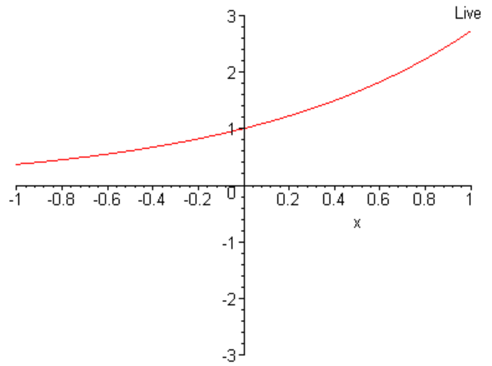
The graph of f(x) is sketched below. Find d.



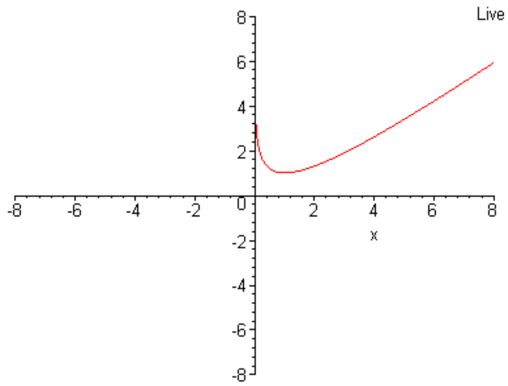
Task 9. Below you see the graphs of f(x) and g(x). Sketch the graph of the function f(x) – g(x). You don't need to put more numbers on the axis. A rough sketch is enough.



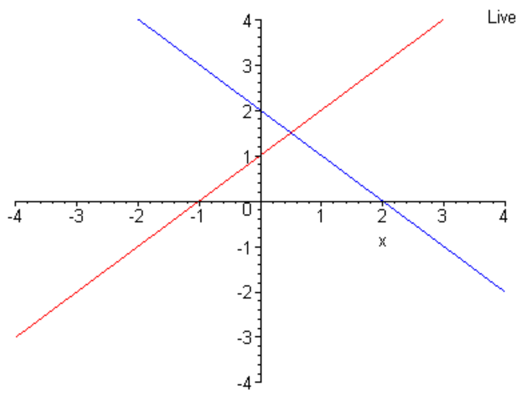
Task 10. The graph of f(x) is shown below. Sketch the graph of f(-x). You don't need to put more numbers on the axis. A rough sketch is enough.



Task 11. The graph of f(x) is shown below. Sketch the graph of -f(x). You don't need to put more numbers on the axis. A rough sketch is enough.



Task 12. The graphs of two functions are shown below. Sketch the graph of the sum of the two functions. You don't need to put more numbers on the axis. A rough sketch is enough.



Task 13. The cost of producing x units of a product is given by $K(x) = -0,1x^2 + 6x + 200$ when x is in the interval [0,20]. Calculate the marginal cost $K'(x)$.

Estimate the marginal cost for x=10. What is the interpretation of this number? Estimate the marginal cost for x=15. What is the interpretation of this number in relation to the answer you got in the previous question?

Task 14. Suppose f(x) is a function of third degree and that g(x) is a linear function. What kind of function is $h(x) = f(x) \cdot g(x)$?

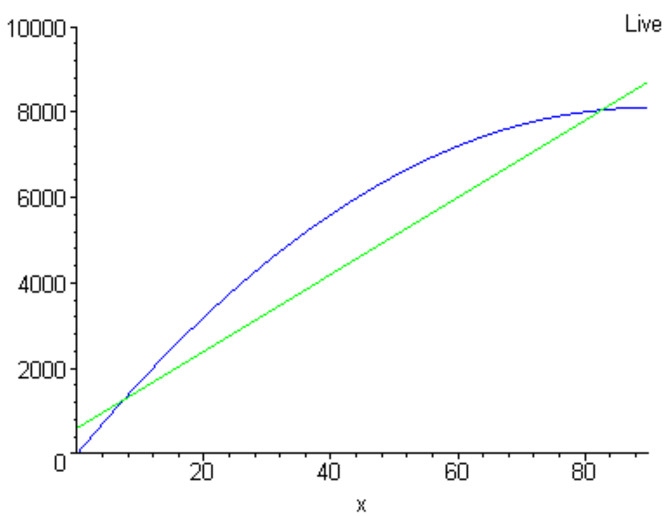
Task 15. Suppose f(x) is a function of third degree and that g(x) is a function of second degree and that f(x) can be divided by g(x). What kind of function is $j(x) = f(x)/g(x)$?

Task 16. Calculate the derivative:

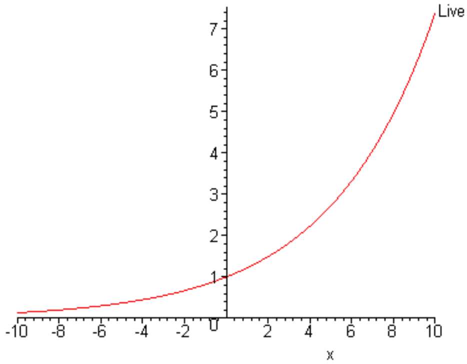
a) $f(x) = 4x + 2$ b) $g(x) = 3x^4 + x^2 - 6x + 4$ c) $h(x) = \dfrac{x^2 - 2x}{2x + 4}$

d) $m(x) = \ln(3x)$ e) $n(x) = e^{ax+b}$ where a and b are constants

Task 17. The graphs of two functions are shown below. The linear function is a cost function giving the total cost by producing x units of a product and is given by $K(x) = 600 + 90x$. The other graph shows the total income by selling x units of the product. How much is the marginal income when the profit is at its maximum?



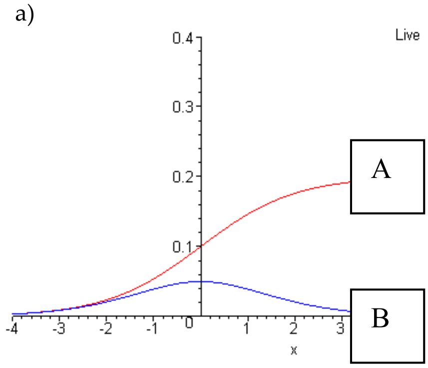
Task 18. Use the graph below to decide when f(x) > 2. You only need to write the answer.



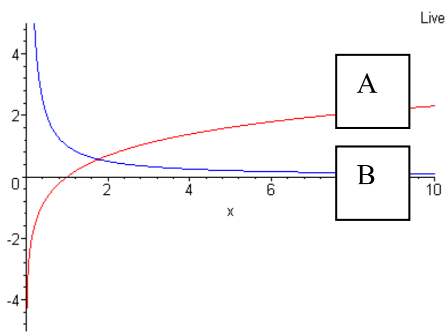
Task 19. A company has a linear cost function. The cost of producing 15 units is 605 and the cost of producing 31 units is 877. What is the cost of producing 8 units?

Task 20. The graph of a function f(x) and its derivative is shown in the same coordinate system. Decide whether A or B is the derivative.

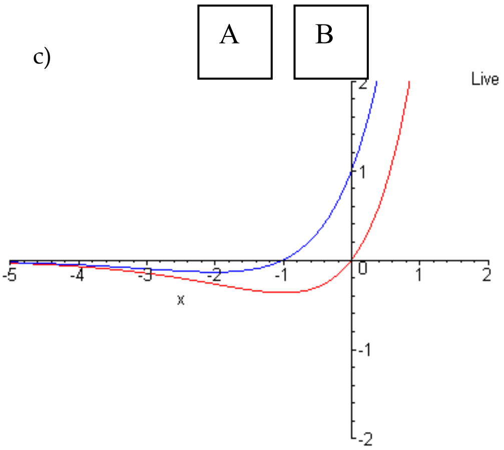
a)



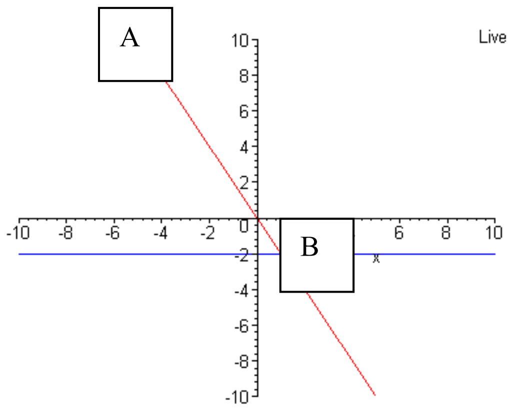
b)



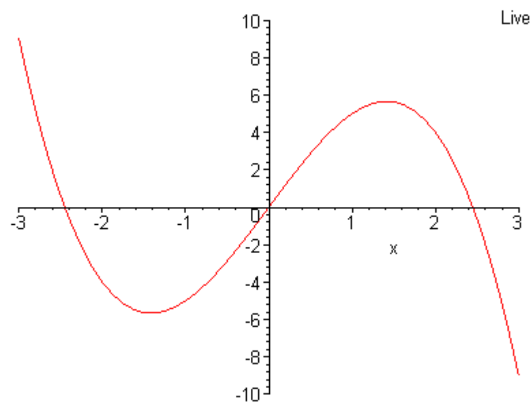
c)



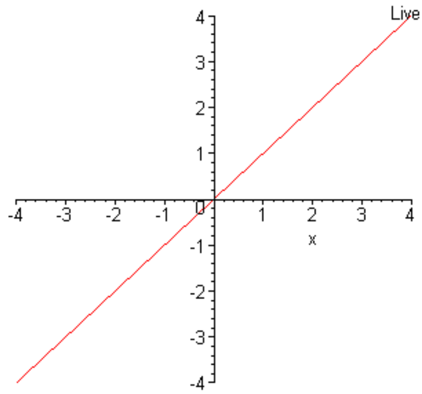
d)



Task 21. The graph of the function h(x) is shown below. Fill in a schema for the sign of h′(x). When is h′(x) largest?



Task 22. Let the function f(x) be given by $f(x) = x^2 + 2x$. Suppose g(x) is given by the graph below. What is the expression for $h(x) = f(x) \cdot g(x)$?



Task 23. g(x) is a linear function. Write down the expression for g(x) when g(2)=0 and g(0)=4.

Appendix B

Structural equation modelling (SEM)

STRUCTURAL EQUATION MODELING IN GENERAL

Concepts are linked to observable variables that are normally measured by questionnaire items. In Figure B-1 *item1, item2* and *item3* are observable, measurable variables measured by scores on tasks, while the latent variable represents the concept:

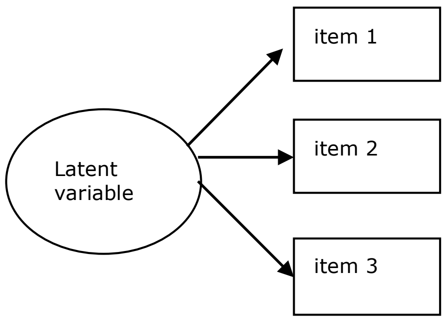


Figure B-2. The measurement model for a construct represented by a latent variable.

The direction of the arrows suggests a causal relationship, i.e. the latent variables are assumed to have an effect on the item scores. Research question 1 is illustrated in path diagrams like the one in Figure B-1. All latent variables are measured in a similar manner and this represents the measurement part of a model that can also be described by a set of equations.

The structural part of the model concerns relations between several latent variables connected through linear regression equations where latent variables serve as dependent and independent variables in the regression.

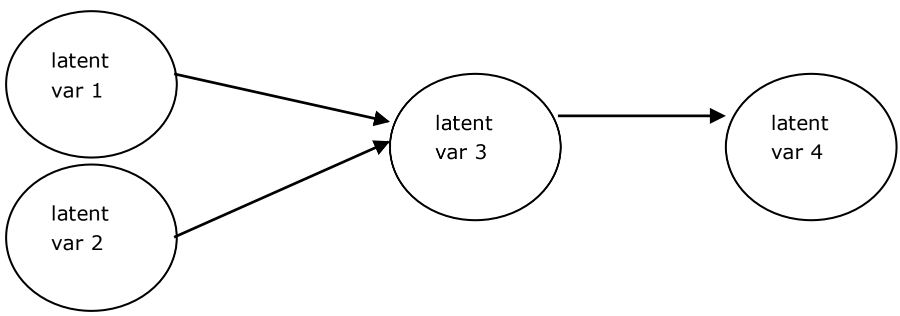


Figure B-2. The latent variable model represented as a path diagram.

Each arrow points from a latent variable being the independent variable to a dependent variable in a linear regression. In the example described in Figure B-2, the latent variables *var 1* and *var 2* are denoted as exogenous as their causes lies outside the model, while the variables *var 3* and *var 4* are endogenous variables, as they are determined by variables within the model (Bollen, 1989, p. 12). Research questions 2 and 3 are represented by a path diagram with latent variables. The ability to apply functions and conceptual knowledge of functions are treated as endogenous, while procedural knowledge of functions is seen as exogenous.

An advantage with structural equation modelling is that it allows us to study both the measurement model (factor analysis model) and the latent variable model (regression equation models) within the frame of one single model. The model emphasizes covariances or correlations between the observed indicators rather than individual cases. Instead of minimizing the functions of observed and predicted individual values, as in traditional regression analysis, we minimize the difference between the sample covariances (or correlations) and the covariances (or correlations) predicted by the model.

As an example, assume that we have hypothesized a model as shown in Figure B-3. The items are represented by rectangles and the factors by ellipses. Assume further that *item a* and *item b* have an observed correlation. In a similar manner, correlations are observed for all pairs of items. Hence we observe a matrix of correlations. The model would result in a 9x9 correlation matrix with 36 observed correlations[32].

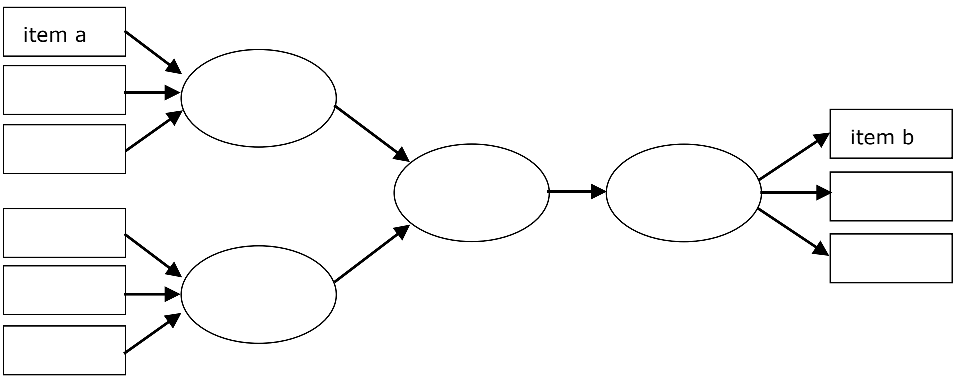


Figure B-3. The complete structural equation model consisting of the measurement model and the latent variable model.

Factor loadings and regression parameters are estimated for each arrow in the model, and are used to compute the correlation between *item a* and *item b* as it would be according to the estimated model. The same is done for all pairs of items, and a model estimated matrix of all correlations is computed. In this study all three research questions are analyzed simultaneously in one structural equation model that will be illustrated in a path diagram similar to the model in Figure B-3.

This means that a matrix with observed covariances between the items is compared to the covariances predicted by the model. Small differences between the observed correlations and the model-estimated correlations indicates good fit, i.e. the estimated

[32] Each correlation occurs in pairs and the elements on the diagonal are all equal to 1.

model fits the data well. Otherwise, if there are large discrepancies between the observed and the model generated covariance matrixes, the model is not good.

Several methods for estimation of parameters and model fit are discussed in connection with the estimation. It could be criticized that the observed correlations are used to estimate parameters and thereafter used to determine the goodness of fit. This is the same phenomenon that we find in linear regression analysis, where the observed data are used to estimate the regression parameters, and then to calculate fit in terms of the R-square. However, the fit measures estimated this way seems to be accepted.

Structural equation modelling (SEM) consists of several components and includes several statistical techniques. A number of statistical analysis methods and program packages are available. The analysis in this dissertation is developed by use of the software package LISREL (linear structural relations) (Jöreskog & Sörbom, 1993, Jöreskog, 1970) that makes it possible to study relationships between latent variables using regression techniques in combination with factor analysis to indirectly measure the latent variables via observable variables.

FACTOR ANALYSIS

Factor analysis is a set of methods, often used in social and behavioural sciences, used to group together several variables that are related as one common factor. The factor analysis emphasizes the relation of latent factors to observed variables (Spearman, 1904). In the measurement model, the observed variables, also called items, are regarded as the effects of the latent variables (Bollen, 1989, p. 7). For example, let's suppose we want to study customer satisfaction related to a grocery store. One approach would be to let the customers answer *yes* or *no* to the question of whether they consider themselves as satisfied or not, but this procedure would not cover the different aspects of what is understood by customer satisfaction. It would only put the answers in two different categories, "satisfied" or "not satisfied". Instead, we could specify customer satisfaction as satisfaction with service, price level, availability and product quality, and try to measure each of them on an interval scale. If these four aspects reflect what we mean by customer satisfaction, we would expect people who answer *yes* on customer satisfaction to give a high score on the four items and vice versa. In the terminology of factor analysis, the four items are being represented by four observable variables, regarded as effects of the common factor, customer satisfaction. In this way customer satisfaction is not measured directly, but indirectly through the items. In the same way, it would be incorrect to use a dichotomous variable to measure conceptual knowledge of functions. The distinction between exploratory factor analysis (EFA) and confirmatory factor analysis (CFA) is important for this study. Exploratory factor analysis is used to detect a factor structure, given a set of variables, while confirmatory factor analysis is used to test whether a set of data supports a given factor structure. The items in this study are grouped together based on an assumption of an underlying factor structure. Therefore the aim is to study whether the data supports this a priori factor structure, and consequently confirmatory factor analysis was the approach here.

If we look at one single factor, the analysis estimates one factor loading (λ_i) for each item. It is possible for an item to load on several factors, but that is not the case in the present study. In this way, the confirmatory factor analysis model focuses on the linear relationship between factors and their measured variables. For example, the dependence of the variable x_i on the factor ξ (xi) as illustrated in Figure B-4.

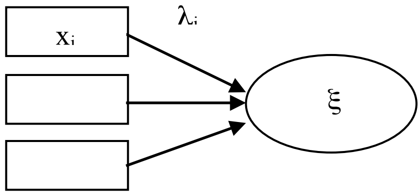


Figure B-4. Each item (x_i) loads on the factor (ξ) with a factor loading (λ_i).

The relationship between x_i and ξ can also be represented by an equation shown in equation B.1 below.

$$x_i = \lambda_i \xi + \delta_I \qquad \text{(B.1)}$$

where δ_i is the random disturbance term. Hence, the intention of the confirmatory factor analysis is to estimate sound scores for all factor loadings in the model.

LATENT VARIABLE MODEL

The concepts in structural equation models are represented by latent variables. The analysis does not determine causal relations between the latent variables, but determines to what extent they are related to each other.

A latent variable can be more or less directly measurable. A concept like depression is obviously not directly measurable, while variables such as working experience may be measurable. In other words, whether a variable is directly measurable or not may vary in degree, but analytically they are treated the same (Bollen, 1989, p. 11).

The latent variable model, describes the linear relations between the latent variables in terms of linear regression equations. Given a set of latent variables, several combinations of relations are possible, and the suggested model is often best illustrated using a path diagram with arrows and ellipses. The ellipses represent latent variables, and the arrows represent influences between the variables. Let A, B, and C be three latent variables. Figure B-5 suggests two different models connecting the three variables:

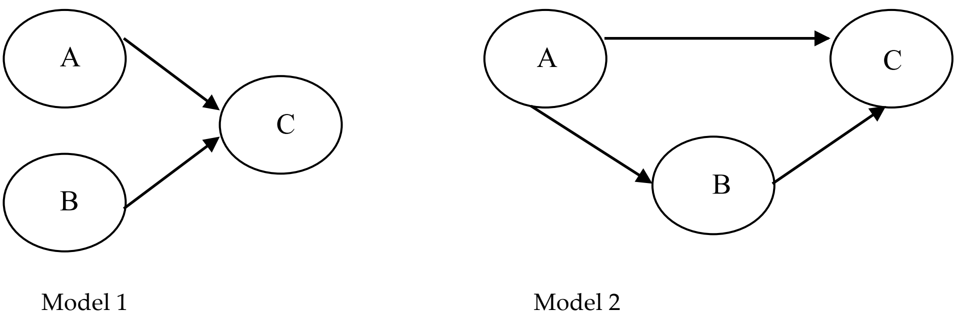


Figure B-5. The same set of factors may be used in different model structures.

In model 1, A and B are exogenous and C endogenous, while A is exogenous with B and C as endogenous in model 2. A further look at model 2 suggests that the effect A has on C in model 2 can be separated in two types, a direct and an indirect effect. A has a direct effect on C and an indirect effect on C through an intermediating variable, B. The total effect from A on C is the sum of the direct and indirect effect.

Each linear equation is defined by means of parameters. In general, the number of parameters to be estimated cannot exceed the number of variables, so even if models can be defined with bi-directional arrows, model 2 would be under-identified. Model 2, as it is shown in Figure B-5, is just-identified, while model 1 is over-identified with three variables and two parameters.

Publications of the University of Eastern Finland

Dissertations in Education, Humanities, and Theology

Taru Viinikainen. *Taipuuko "akrobaatti Aleksandra"? Nimikekonstruktio ja nimikkeen taipuminen lehtikielessä 1900-luvulta 2000-luvulle*. 2010.
Pekka Metso. *Divine Presence in the Eucharistic Theology of Nicholas Cabasilas*. 2010.
Pekka Kilpeläinen. *In Search of a Postcategorical Utopia. James Baldwin and the Politics of 'Race' and Sexuality*. 2010.
Leena Vartiainen. *Yhteisöllinen käsityö. Verkostoja, taitoja ja yhteisiä elämyksiä*. 2010.
Alexandra Simon-López. *Hypersurrealism. Surrealist Literary Hypertexts*. 2010.
Merja Sagulin. *Jälkiä ajan hiekassa. Kontekstuaalinen tutkimus Daniel Defoen Robinson Crusoen suomenkielisten adaptaatioiden aatteellisista ja kirjallisista traditioista sekä subjektikäsityksistä*. 2010.
Pirkko Pollari. *Vapaan sivistystyön kieltenopettajien pedagogiset ratkaisut ja käytänteet teknologiaa hyödyntävässä vieraiden kielten opetuksessa*. 2010.
Ulla Piela. *Kansanparannuksen kerrotut merkitykset Pohjois-Karjalassa 1800- ja 1900-luvuilla*. 2010.
Lea Meriläinen. *Language Transfer in the Written English of Finnish Students.* 2010.
Kati Aho-Mustonen. *Group Psychoeducation for Forensic Long-term Patients with Schizophrenia*. 2011.
Anne-Maria Nupponen. *»Savon murre» savolaiskorvin. Kansa murteen havainnoijana*. 2011.
Teemu Valtonen. *An Insight into Collaborative Learning with ICT: Teachers' and Students' Perspectives*. 2011.
Teemu Kakkuri. *Evankelinen liike kirkossa ja yhteiskunnassa 1944-1963. Aktiivinen uudistusliike ja konservatiivinen sopeutuja*. 2011.
Riitta Kärkkäinen. *Doing Better? Children's and Their Parents' and Teachers' Perceptions of the Malleability of the Child's Academic Competences.* 2011.
Jouko Kiiski. *Suomalainen avioero 2000-luvun alussa. Miksi avioliitto puretaan, miten ero koetaan ja miten siitä selviydytään*. 2011.
Liisa Timonen. *Kansainvälisty tai väisty? Tapaustutkimus kansainvälisyysosaamisen ja kulttuurienvälisen oppimisen merkityksenannoista oppijan, opettajan ja korkeakoulutoimijan pedagogisen suhteen rajaamissa kohtaamisen tiloissa.* 2011.
Matti Vänttinen. *Oikeasti hyvä numero. Oppilaiden arvioinnin totuudet ja totuustuotanto rinnakkaiskoulusta yhtenäiskouluun*. 2011.
Merja Ylönen. *Aikuiset opin poluilla. Oppimistukikeskuksen asiakkaiden opiskelukokemuksista ja kouluttautumishalukkuudelle merkityksellisistä tekijöistä.* 2011.

Kirsi Pankarinkangas. *Leskien keski-iässä tai myöhemmällä iällä solmimat uudet avioliitot. Seurantatutkimus.* 2011.
Olavi Leino. *Oppisopimusopiskelijan oppimisen henkilökohtaistaminen ja oppimismahdollisuudet työpaikalla*. 2011.
Kristiina Abdallah. *Translators in Production Networks. Reflections on Agency, Quality and Ethics*. 2012.
Riina Kokkonen. *Mittarissa lapsen keho ja vanhemmuus – tervettä lasta sekä "hyvää" ja "huonoa" vanhemmuutta koskevia tulkintoja nyky-Suomessa*. 2012.
Ari Sivenius. *Aikuislukion eetos opettajien merkityksenantojen valossa*. 2012.
Kamal Sbiri. *Voices from the Margin. Rethinking History, Identity, and Belonging in the Contemporary North African Anglophone Novel*. 2012.
Ville Sassi. *Uudenlaisen pahan unohdettu historia. Arvohistoriallinen tutkimus 1980-luvun suomalaisen romaanin pahan tematiikasta ja "pahan koulukunta" –vuosikymmenmääritteen muodostumisesta kirjallisuusjärjestelmässä*. 2012.
Merja Hyytiäinen. *Integroiden, segregoiden ja osallistaen. Kolmen vaikeasti kehitysvammaisen oppilaan opiskelu yleisopetuksessa ja koulupolku esiopetuksesta toiselle asteelle*. 2012.
Hanna Mikkola. *"Tänään työ on kauneus on ruumis on laihuus." Feministinen luenta syömishäiriöiden ja naissukupuolen kytköksistä suomalaisissa syömishäiriöromaaneissa*. 2012.
Aino Äikäs. *Toiselta asteelta eteenpäin. Narratiivinen tutkimus vaikeavammaisen nuoren aikuisen koulutuksesta ja työllistymisestä*. 2012.
Maija Korhonen. *Yrittäjyyttä ja yrittäjämäisyyttä kaikille? Uusliberalistinen hallinta, koulutettavuus ja sosiaaliset erot peruskoulun yrittäjyyskasvatuksessa*. 2012.
Päivikki Ronkainen. *Yhteinen tehtävä. Muutoksen avaama kehittämispyrkimys opettajayhteisössä*. 2012.
Kalevi Paldanius. *Eläinlääkärin ammatti-identiteetti, asiakasvuorovaikutuksen jännitteiden hallinta ja kliinisen päättelyn yhteenkietoutuminen sekapraktiikassa*. 2012
Kari Korolainen. *Koristelun kuvailu. Kategorisoinnin analyysi*. 2012.
Maija Metsämäki. *Influencing through Language. Studies in L2 Debate*. 2012.
Pål Lauritzen. *Conceptual and Procedural Knowledge of Mathematical Functions*. 2012.

Pål Lauritzen

Conceptual and Procedural Knowledge of Mathematical Functions

Function is one of the most important concepts and tools in mathematics. This thesis discusses it as a complex of conceptual and procedural knowledge and suggests viable and sustainable educational practices for mathematics teaching based on empirical research.

UNIVERSITY OF EASTERN FINLAND

Publications of the University of Eastern Finland
Dissertations in Education, Humanities, and Theology

isbn: 978-952-61-0892-6 (print)
issnl: 1798-5625
issn: 1798-5625
isbn: 978-952-61-0893-3 (pdf)
issn: 1798-5633