QUANTUM ANALYSIS OF YOUNG'S INTERFERENCE EXPERIMENT FOR ELECTROMAGNETIC FIELDS

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Abstract

In this work, we formulate a quantum–mechanical description of interference of electromagnetic fields in Young's interference experiment, thereby taking into account the polarization properties of the field and describing them in terms of quantum analogs of classical Stokes parameters. Commencing with the classical theory of interference of scalar fields, we proceed to a relatively advanced approach to electromagnetic interference, bringing into the equation cross–spectral density tensor, polarization matrix and Stokes parameters to analyze the polarization properties. Subsequently, the same phenomenon is analyzed in the domain of quantum optics, thereby expressing the fields as operators and observables as the expectation values. Firstly, an outline of the scalar approach of the operators in the interference experiment is presented to establish the foundation to base the electromagnetic approach on, followed by a full description of quantum analog of electromagnetic interference in Young's experiment. In particular, Stokes parameters are adopted to calculate the polarization effects in quantum theory.

Preface

This report attempts to condense and present the theories related to the interference in the canonical Young's double pinhole experiment, prior to heading to investigate the phenomenon with the quantum analysis of electromagnetic fields. My interest in the realm of quantum mechanics has encouraged me to work on this thesis, wherefrom I have learnt a lot during these times.

I am really grateful to my supervisors, Prof. Ari T. Friberg and Prof. Jani Tervo for guiding me and believing in me. I will always be indebted for their constant support and confidence on me. I would also like to offer my special thanks to the Faculty of Forestry and Sciences for providing the financial assistance during my Master's studies.

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Joensuu, the 28th of October 2014 *Anjan Shrestha*

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Chapter I

Introduction

From classical optics, light can be considered as an electromagnetic field with its constituents, electric field and magnetic field propagating in unison through a medium. For the sake of convenience, we reasonably assume the field to be deterministic, i.e., the disturbance caused by the field is predictable at any point in space and time. However, in reality any field has an inherent randomness in it, which could be attributed to random fluctuations of light sources or the medium through which light propagates [1]. Essentially, generation of light occurs due to the atomic emissions; as the electrons undergo quantum jumps, with the transition occuring after a minuscule duration of about 10 ns, they emit spontaneously a wavetrain and superposition of these wavetrains emanating independently at different frequencies and phases from a very large number of atoms results in the randomness of light [2]. In addition, the randomness may also be variations to the optical wavefront caused by scattering from a rough surface, diffused glass, or turbulent fluids. This study of the random fluctuation of light and its effects falls under the theory of optical coherence [1].

Conventionally, the study of coherence was limited to the scalar approximations of the light field, however, interests towards the electromagnetic coherence theory increased with the development of subwavelength nanostructures. Such structures give rise to near-field coherence phenomena, e.g., surface plasmons, that the scalar coherence theory is generally unable to model rigorously.

The interference experiment typified by Young's interference experiment has played a central role to understand the coherence of the field. Classical theory, which is based on the wave nature of light, could conveniently describe the interference pattern, however the first study of interference based on the quantum nature

of light was done by Dirac [3]; his work took into account the scalar description of the fields and in this thesis, we extend the concepts to electromagnetic fields.

In this thesis, we present the quantum analysis of interference of electromagnetic fields. Beginning with the preliminary knowledge of correlation and polarization in Chapter II, which may prove useful to understand the forthcoming concepts, we move on to lay out the classical scalar theory of interference and introduce the coherence concepts in Chapter III, followed with the electromagnetic approach in the classical domain in Chapter IV. In the following chapters, we redirect our attention towards the quantum domain, introducing the quantum–mechanical first–order coherence functions and presenting a quantum formulation of Young's experiment for scalar fields in Chapter V and extending these concepts to incorporate electromagnetic fields to formulate the quantum interference law in Chapter VI. Finally in Chapter VII, we summarize and discuss about the results.

Chapter II

Preliminaries

In this chapter, we cover the fundamental concepts needed to reasonably understand the theories involving the optical coherence and the interference of the waves and the quantum description of the relevant phenomena.

2.1 Statistical concepts

Real waves are never completely coherent or incoherent; these conditions are more of conceptual idealizations than physical reality. In fact, any wave suffers from randomness, accounted to the random emission of the wavetrain itself and the fluctuations of the transmitting media. As a consequence, the phase and amplitude of the wave fluctuate randomly in space and time. However, some meaningful properties could be extracted from the randomness by performing statistical analysis of the field, which characterizes and distinguishes it from the other fields. In the following statistical approach, we assume scalar description of light, i.e., the lightwaves propagate paraxially and are elliptically polarized [1, 2].

2.1.1 Probability density, expectation value, and time averages

Although all the field quantities are real-valued, it is customary to employ the complex field representation to ease the mathematical analysis. For the sake of convenience, we ignore here the position dependence of the wave by considering its disturbance with time at a certain point in space, thereby denoting the wave by its complex analytic signal $U(t)$ [1]. Since $U(t)$ is a random function in time, its values form a distribution in the complex plane; the distribution is governed by the probability distribution $p_1(U, t)$ where the subscript 1 denotes one-fold probability.

The probability density is time-dependent and since there is always some value at every instant t, we have

$$\int_C p_1(U,t)\,\mathrm{d}U = 1, \tag{2.1}$$

where the integration is performed over the complex plane C [4]. The expectation value of $U(t)$ at time t is defined by

$$\langle U(t)\rangle = \int_C p_1(U,t)U\,\mathrm{d}U. \tag{2.2}$$

The expectation value of $U(t)$ can also be expressed in terms of ensemble average; the random function $U(t)$ can have infinite set of possible values, called realizations $U_1, U_2, \ldots$ known as statistical ensemble whose average is given by

$$\langle U(t)\rangle = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} U_n(t). \tag{2.3}$$

Though one-fold probability density is very helpful to determine the expectation value of a function at any arbitrary time, it manifests no information about the possible correlations between the functions at two different times t_1 and t_2. The information about this connection is described by the joint or two–fold probability density $p_2(U_1,t_1;U_2,t_2)$ where the subscript 2 denotes two–fold probability density. Analogously to the one–fold probability density, p_2 obeys the normalization property [4]

$$\int_C\int_C p_2(U_1,t_1;U_2,t_2)\,\mathrm{d}U_1\mathrm{d}U_2 = 1. \tag{2.4}$$

Thus, there exists an infinite hierarchy of probability densities, p_1, p_2, p_3, ... each containing all the information contained in the previous ones. [4]

2.1.2 Correlation functions

Despite the randomness of the field, the fields at two instants of time, space or both may fluctuate in complete harmony or have no relation whatsoever, depending upon how close in space or time domain the measurements are taken [1]. This means of comparing the signals to determine the degree of similarity falls on the realm of correlation analysis, classified as autocorrelation or cross–correlation functions. [2].

The (two-time) autocorrelation function of $U(t)$ at two instants of time, t_1 and t_2 is given by [2]

$$\Gamma(t_1, t_2) = \langle U^*(t_1)U(t_2)\rangle = \int_C \int_C U_1^* U_2 p_2(U_1, t_1; U_2, t_2)\, \mathrm{d}U_1 \mathrm{d}U_2. \tag{2.5}$$

There also exists higher-order correlation functions following higher probability densities that contain more information than the previous ones, for instance, a fourth-order correlation function could reveal the information about the intensity correlations. However, we limit ourselves to second–order correlation functions to examine the coherence in Young's experiment. Autocorrelation function is Hermitian, i.e.,

$$\Gamma(t_1, t_2) = \Gamma^*(t_2, t_1). \tag{2.6}$$

Often we are interested in the spatiotemporal behaviour of a random field $U(\boldsymbol{r}, t)$. The correlation properties of such a field are described by the cross–correlation function [4]

$$\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2, t_1, t_2) = \langle U^*(\boldsymbol{r}_1, t_1)U(\boldsymbol{r}_2, t_2)\rangle. \tag{2.7}$$

2.1.3 Stationarity and ergodicity

Though the field is time–dependent, its statistical properties may well be invariant of time, i.e., the character of fluctuations remains the same. In other words, all the probability densities p_1, p_2,... remain invariant under arbitrary translation of the origin of time and consequently the expectation value. Furthermore the measurable property of the field intensity, given by the ensemble average of the absolute square of the field also remains constant with time. Therefore, [4]

$$p_n(U_n, t_n; U_{n-1}, t_{n-1}...; U_1, t_1) = p_n(U_n, t_n + T; U_{n-1}, t_{n-1} + T, ...U_1, t_1 + T), \tag{2.8}$$

$$\langle U(t_1), U(t_2), ...\rangle = \langle U(t_1 + T), U(t_2 + T), ...\rangle, \tag{2.9}$$

where T is an arbitrary time interval. Such a field is called statistically stationary field. Clearly, stationarity should not be mistaken for constancy in the field but constancy in the average properties of the field. Examples of stationary field include thermal light, continuous lasers beams, etc [1]. In classical coherence theory, higher–order correlation functions are uncommon and therefore, we define a field with stationarity to the mean value and second–order correlation functions as stationary in the wide sense. For a stationary field, the time average for a particular

realization $U_n(t)$ is determined by averaging the field over infinitely long interval, given by [1]

$$\overline{U}_n = \lim_{T\to\infty} \frac{1}{T} \int_{t-T/2}^{t+T/2} U_n(t)\, \mathrm{d}t, \tag{2.10}$$

which is independent of T or t but depends on the particular realization n of the ensemble.

Ergodicity describes a statistical property of a random function when all realizations have the same statistical parameters [5], thus the time averages of the realizations are equal and same as the ensemble average. Often when the field is stationary, it exhibits ergodicity. Therefore, for an ergodic field the averaging could be performed over realizations or over time, with the same result. We assume the field to be statistically stationary and ergodic throughout this thesis. As the time dependence vanishes for statistically stationary ergodic fields, the correlation analysis remains indifferent to the time instants taken but depend solely on the time delay between them, $\tau = t_2 - t_1$ and is defined as

$$\Gamma(t_1, t_2) = \Gamma(\tau), \tag{2.11}$$

$$\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2, t_1, t_2) = \Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2, \tau). \tag{2.12}$$

2.2 Coherence concepts

The coherence properties of a field are usually described in terms of second–order correlation functions [4]. In the language of optical coherence theory, the autocorrelation function of a random stationary ergodic function $\Gamma(\tau)$, Eq. (2.10) is called the temporal coherence function, which equals the intensity I when $\tau = 0$, i.e.,

$$\Gamma(0) = \langle U^*(t)U(t)\rangle = I \tag{2.13}$$

A measure of coherence of the field without carrying information about the intensity is given by the normalized version of the temporal coherence function, called the complex degree of temporal coherence.

$$\gamma(\tau) = \frac{\Gamma(\tau)}{\Gamma(0)} = \frac{\langle U^*(t)U(t+\tau)\rangle}{\langle U^*(t)U(t)\rangle} \tag{2.14}$$

From Schwarz inequality, it can be shown that the absolute value lies between 0 and 1, i.e., $0 \leq \gamma(\tau) \leq 1$ where $\gamma(\tau) = 1$ stands for complete correlation and vice versa.

Likewise, the cross-correlation, which describes the relation between the temporal and spatial fluctuations of a random function $U(t)$ is called the mutual coherence function whereas its normalized version is called the complex degree of coherence $\gamma(\boldsymbol{r}_1, \boldsymbol{r}_2, \tau)$. For a stationary field, we can write from Eq. (2.12) [1, 2]

$$\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2, \tau) = \langle U^*(\boldsymbol{r}_1, t) U(\boldsymbol{r}_2, t+\tau) \rangle \tag{2.15}$$

$$\gamma(\boldsymbol{r}_1, \boldsymbol{r}_2, \tau) = \frac{\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2, \tau)}{[\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_1, 0)\Gamma(\boldsymbol{r}_2, \boldsymbol{r}_2, 0)]^{1/2}} \tag{2.16}$$

Analogously to the complex degree of temporal coherence, complex degree of coherence also has its absolute value in the limit $0 \leq |\gamma(\boldsymbol{r}_1, \boldsymbol{r}_2, \tau)| \leq 1$ such that $|\gamma(\boldsymbol{r}_1, \boldsymbol{r}_2, \tau)|$ takes the value 0 or 1 when the fluctuations at $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ at a time delay of τ are completely uncorrelated or completely correlated respectively, i.e., completely incoherent or coherent field respectively. The domain of partial coherence exists in the region of $0 < |\gamma(\boldsymbol{r}_1, \boldsymbol{r}_2, \tau)| < 1$ [1]. It should be noted however that $|\gamma(\boldsymbol{r}_1, \boldsymbol{r}_2, \tau)|$ equals 1 for all values of τ and for all pair of spatial points only if the field is perfectly monochromatic, an idealization of the practical field. Likewise, $|\gamma(\boldsymbol{r}_1, \boldsymbol{r}_2, \tau)| = 0$ for all pair of points with any time delay τ cannot exist for a non–zero radiation field either, which conclude essentially that the real fields are always partially coherent, rather than being the extremes at each end [2].

An alternative approach to the space–time domain for examining the coherence effects is the space–frequency domain, which is more desirable since most materials are strongly dispersive in the optical frequencies.

The power spectral density, or spectral density, or simply the spectrum $S(\omega)$ is defined as the Fourier transform of the temporal coherence function: [4]

$$S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Gamma(\tau) \exp(i\omega\tau) \, \mathrm{d}\tau, \tag{2.17}$$

whereas

$$\Gamma(\tau) = \int_{0}^{\infty} S(\omega) \exp(-i\omega t) \, \mathrm{d}\omega. \tag{2.18}$$

This relation is known as Wiener–Khintchine theorem [1, 2]. Likewise, the Fourier transform of the mutual coherence function $\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2, \tau)$, called the cross spectral density $W(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega)$, which should not be mistaken as a measure of spatial coherence between points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ at the angular frequency ω; it turns out to be a correlation

between complex random functions, discussed further in the next chapter. Thus, the function becomes

$$W(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2, \tau) \exp(i\omega\tau)\, d\tau. \tag{2.19}$$

Analogously to Eq. (2.16), the normalized version of cross spectral density is written as

$$\mu(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega) = \frac{W(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega)}{[S(\boldsymbol{r}_1, \omega) \cdot S(\boldsymbol{r}_2, \omega)]^{1/2}} \tag{2.20}$$

where the absolute value, $|\mu(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega)|$ lies within 0 and 1, i.e., $0 \leq \mu(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega) \leq 1$. Here $S(\boldsymbol{r}, \omega) = W(\boldsymbol{r}, \boldsymbol{r}, \omega)$ is the spectral density at position $\boldsymbol{r}$ and at frequency ω.

The correlation between the fluctuations of a random function $U(t)$ at two instants of time is described by the complex degree of coherence $\gamma(\tau)$, which usually decreases as τ increases. If $|\gamma(\tau)|$ decreases monotonically, then the width of the distribution at which $|\gamma(\tau)|$ lowers to a certain value is called the coherence time of the field τ_c. Likewise, the coherence length l_c is defined as [1]

$$l_c = c\tau_c. \tag{2.21}$$

The spectral width, or bandwidth $\Delta\omega$ is defined as the width of the spectral density. Since the spectral density and the temporal coherence function are Fourier transforms of each other, the bandwidth is inversely proportional to the coherence time. However, the fundamental definition of the width could be established in several ways depending on the spectral profile. [1]

An important parameter that characterizes the random light is the coherence area A_c. Essentially, it is the cross-sectional area of the $|\gamma(\boldsymbol{r}_1, \boldsymbol{r}_2, 0)|$ distribution about any point $\boldsymbol{r}$ taken at the height when $|\gamma(\boldsymbol{r}_1, \boldsymbol{r}_2, 0)|$ drops to a prescribed value as $|\boldsymbol{r}_1 - \boldsymbol{r}_2|$ increases [1]. The coherent area of the field is of considerable interest when it interacts with optical system with apertures; if the area is larger than the size of the aperture, the transmitted field may be regarded as coherent.

Coherence can be, conveniently, classified as spatial or temporal coherence based on whether the correlation is investigated between points in space or instants of time. Spatial coherence is a measure of correlation between fluctuations at two points in space; it relates directly to the finite spatial extent of ordinary light source in space. Temporal coherence relates directly to the finite bandwidth, and therefore, finite coherence time of the source. It describes the correlation between fluctuations

of a point in space at any two instants in time; the fluctuations would be highly correlated if the time interval is less than the coherence time [2].

2.3 Polarization concepts

Polarization is a property associated with waves that can oscillate in more than one direction. In optics, polarization of the field refers specifically to the direction of the electric field [1,2]. Polarization of light is a crucial parameter in some measurement techniques and has found ever–increasing applications in the field of engineering, geology, ellipsometry, and astronomy. Some common applications involve polarized sunglasses, 3D glasses, radio transmission, or display technologies.

A deterministic monochromatic field is always elliptically polarized; the electric field changes its direction or magnitude, or both in a predictable way, either in a linear, circular, or elliptical fashion with the first two being specific cases of the elliptical polarization. The shape and orientation of the ellipse, also referred to as the polarization ellipse defines the polarization state of the field, that could be parameterized in terms of the phase difference $\varepsilon = \varepsilon_y - \varepsilon_x$ and the amplitude ratio $r = a_y/a_x$ or more commonly in terms of the orientation angle φ and the ellipticity angle χ, where the Cartesian components of the field E propagating in the $z-$direction are defined as

$$E_x = a_x \exp[i(kz - \omega t + \varepsilon_x)], \tag{2.22}$$

$$E_y = a_y \exp[i(kz - \omega t + \varepsilon_y)]. \tag{2.23}$$

The orientation angle φ and the ellipticity angle χ, as illustrated in Figure 2.1, are defined in terms of the phase difference ε and the amplitude ratio r as [1]

$$\tan 2\varphi = \frac{2r}{1 - r^2} \cos \varepsilon, \tag{2.24}$$

$$\sin 2\chi = \frac{2r}{1 + r^2} \sin \varepsilon. \tag{2.25}$$

An alternative convenient way to express the polarization properties of a field is the Stokes parameters, a set of four values that describe the polarization in terms of intensity, degree of polarization, and angles of the polarization ellipse. They are

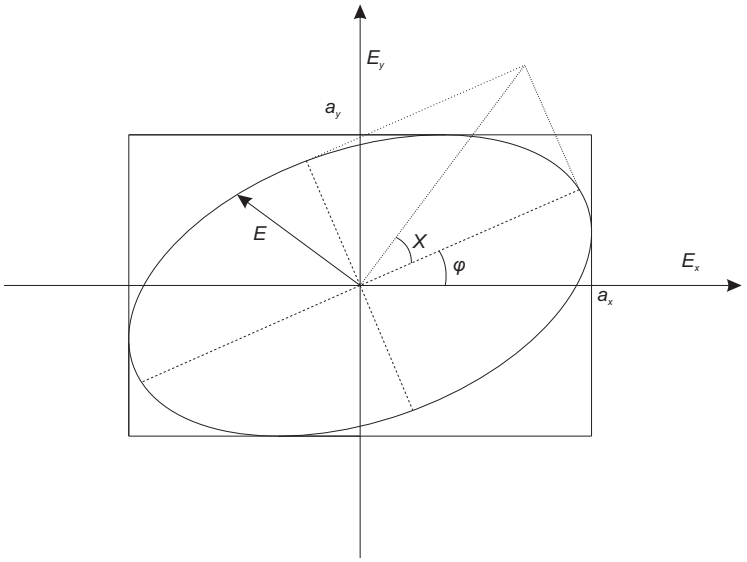

Figure 2.1: Parameterizations of elliptical light. [1]

written as

$$\begin{aligned} S_0 &= I = \langle |E_x|^2 \rangle + \langle |E_y|^2 \rangle, \\ S_1 &= pI \cos 2\varphi \cos 2\chi = \langle |E_x|^2 \rangle - \langle |E_y|^2 \rangle, \\ S_2 &= pI \sin 2\varphi \cos 2\chi = 2\mathrm{Re}\{\langle E_x E_y^* \rangle\}, \\ S_3 &= pI \sin 2\chi = -2\mathrm{Im}\{\langle E_x E_y^* \rangle\}, \end{aligned} \tag{2.26}$$

where I is the total intensity and p is the degree of polarization that describes the polarized portion of the total field. In the physical sense, the Stokes parameters could be interpreted as follows: the first parameter S_0 simply describes the total intensity; the second parameter S_1 describes the superiority of linearly horizontally polarized (LHP) light over linearly vertically polarized light (LVP); the third parameter S_2 describes the superiority of linearly polarized light at $+45°$ over linearly polarized light at $-45°$ and the last value S_3 describes the superiority of right circularly polarized light (RCP) over left circularly polarized (LCP) part [6].

Throughout this thesis, we will employ Stokes parameters to describe the polarization of light since it relies on operational concepts and therefore, could be adopted in quantum physics [7].

Chapter III

Classical scalar theory of coherence

3.1 Coherence in the space–time domain

Based on the assumption that light propagates in the form of waves, classical optics has been successful in explaining different phenomena such as interference, reflection, diffraction and so on, with some exceptions where a quantum description is sought. In the scalar approach, we, however, consider that the lightwaves are uniformly polarized and travel along the same direction so that the they can be treated as scalar waves. Accordingly, the polarization state of the field is obviously overlooked throughout this approach which would require a full electromagnetic approach otherwise.

In the classical Young's interference experiment, we have a broad, statistically stationary light source generating a complex field $U(\boldsymbol{r}, t)$ that propagates along the $z-$axis and illuminates an opaque screen A with two pinholes with centers at point S_1 and S_2, placed orthogonally to the propagation direction as illustrated in Figure 3.1. The pinholes are assumed to be large enough that the diffraction effects inside a pinhole can be neglected yet so small that the field in each can be treated as uniform. The lightwaves emerging from the pinholes interfere as they propagate and fall on the screen B located far away from A. Let $U(\boldsymbol{S}_1, t)$ and $U(\boldsymbol{S}_2, t)$ represent the fields at pinholes at $\mathbf{S}_1$ and $\mathbf{S}_2$ as the original field propagate to them respectively. Intuitively the resultant field at point $\boldsymbol{r}$ on the screen is the superposition of fields emerging from the pinholes and is given by [1, 2]

$$U(\boldsymbol{r}, t) = K_1 U(\boldsymbol{S}_1, t - t_1) + K_2 U(\boldsymbol{S}_2, t - t_2), \tag{3.1}$$

where $t_1 = r_1/c$, $t_2 = r_2/c$, and K_1 and K_2 are complex constants called the propa-

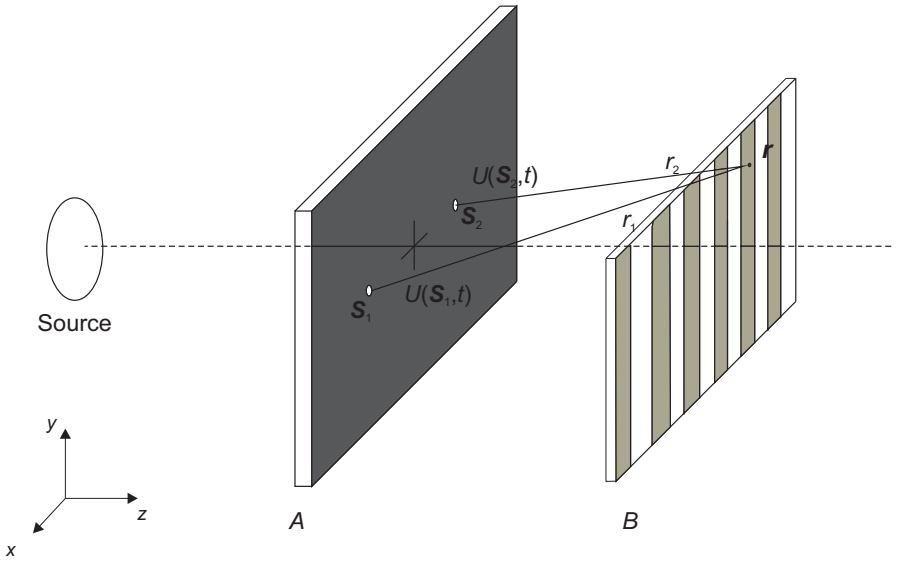

Figure 3.1: Young's two-pinhole interference experiment.

gation factors that depend on the properties of the pinholes and their geometry [4,8]. Mathematically, they alter the field as it emerges out of the pinholes, a phase shift for instance [2]. Since the field is assumed to be stationary and ergodic, the intensity of the resultant field at screen B takes on the form

$$I(\boldsymbol{r}) = I_1 + I_2 + 2\sqrt{I_1 I_2}\ \mathrm{Re}\{\gamma(\boldsymbol{S}_1, \boldsymbol{S}_2, \tau)\}, \tag{3.2}$$

where I_1 and I_2 are the intensities at P when only hole at $\mathbf{S}_1$ or $\mathbf{S}_2$ is open respectively, and $\gamma(\boldsymbol{S}_1, \boldsymbol{S}_2, \tau)$ is the complex degree of coherence between the fields at $\mathbf{S}_1$ and $\mathbf{S}_2$ at a delay of $\tau = t_2 - t_1$. Since $\gamma(\boldsymbol{S}_1, \boldsymbol{S}_2, \tau)$ is complex in nature, Eq. (3.2) could be simplified as

$$I(\boldsymbol{r}) = I_1 + I_2 + 2\sqrt{I_1 I_2}\ |\gamma(\boldsymbol{S}_1, \boldsymbol{S}_2, \tau)| \cos\varphi, \tag{3.3}$$

where $\varphi = \arg\{\gamma(\boldsymbol{S}_1, \boldsymbol{S}_2, \tau)\}$ is the phase of $\gamma(\boldsymbol{S}_1, \boldsymbol{S}_2, \tau)$, which accounts for the transverse locations of maxima and minima of the interference fringes due to variation in the time difference τ. This is the general interference law for partially coherent light. The strength of the interference pattern is described by the visibility, also called the contrast of the interference pattern and given by:

$$V = \frac{I_{\max}(\boldsymbol{r}) - I_{\min}(\boldsymbol{r})}{I_{\max}(\boldsymbol{r}) + I_{\min}(\boldsymbol{r})}. \tag{3.4}$$

The maximum and minimum values are obtained by putting $\cos\varphi$ as -1 and 1 in Eq. (3.3). Therefore, the visibility can be expressed as

$$V = \frac{2\sqrt{I_1 I_2}}{I_1 + I_2} |\gamma(\boldsymbol{S}_1, \boldsymbol{S}_2, \tau)|. \tag{3.5}$$

If the intensities of the field from pinholes are equal, i.e., $I_1 = I_2$, we get

$$V = |\gamma(\boldsymbol{S}_1, \boldsymbol{S}_2, \tau)|. \tag{3.6}$$

Thus, the ability of the wave to interfere is governed by the modulus of the complex degree of coherence at from the pinholes with a time delay equal to the difference in propagation times from the pinholes to a particular point, under a condition that the intensities are equal [1, 2].

3.2 Coherence in the space–frequency domain

Alternatively, the concepts of coherence and interference can also be investigated in the space–frequency domain. In this case, we take into account the spectral density of the field at a particular point for a particular frequency, $S(\boldsymbol{r}, \omega)$ rather than the mean intensity at that point, which brings into question the temporal coherence of the field, $\Gamma(\tau)$. Following the analysis in the space–time domain, if τ' be an arbitrary time difference between the resultant field at point $\boldsymbol{r}$ at screen B, given by Eq. (3.1), then the self–coherence function of the field can be written as

$$\Gamma(\boldsymbol{r}, \boldsymbol{r}, \tau') = \langle U^*(\boldsymbol{r}, t) U(\boldsymbol{r}, t + \tau') \rangle. \tag{3.7}$$

Substituting Eq. (3.1) into the above equation and taking Fourier transform on both sides of the result, we get, with the help of Eq. (2.19),

$$\begin{aligned} S(\boldsymbol{r}, \omega) =& |K_1|^2 W(\boldsymbol{S}_1, \boldsymbol{S}_1, \omega) + |K_2|^2 W(\boldsymbol{S}_2, \boldsymbol{S}_2, \omega) \\ &+ 2|K_1||K_2| \operatorname{Re}\{W(\boldsymbol{S}_1, \boldsymbol{S}_2, \omega) \exp(-i\omega\tau)\}, \end{aligned} \tag{3.8}$$

$$= S_1(\boldsymbol{r}, \omega) + S_2(\boldsymbol{r}, \omega) + 2\sqrt{S_1(\boldsymbol{r}, \omega) S_2(\boldsymbol{r}, \omega)} \operatorname{Re}\{\mu(\boldsymbol{S}_1, \boldsymbol{S}_2, \omega) \exp(-i\omega\tau)\}, \tag{3.9}$$

where $S_1(\boldsymbol{r}, \omega)$ and $S_2(\boldsymbol{r}, \omega)$ are the spectral densities when hole 1 or 2 is open at a time and $\mu(\boldsymbol{S}_1, \boldsymbol{S}_2, \omega)$ is the spectral degree of coherence, as defined by Eq. (2.20). This is the spectral interference law, analogous to the general interference law in

Eq. (3.3) with the intensities replaced by the spectral densities and the temporal coherence by the spectral degree of coherence. Likewise, the spectral visibility at the examined frequency is described by $|\mu(\boldsymbol{S}_1, \boldsymbol{S}_2, \omega)|$ provided that $S_1(\boldsymbol{r}, \omega) = S_2(\boldsymbol{r}, \omega)$.

If we consider the interference from an extended quasi–monochromatic light source with θ_{s} as the angle subtended by the source at the pinhole plane, then the interference fringes are visible given $\theta_{\mathrm{s}} < \bar{\lambda}/L$ where $\bar{\lambda}$ stands for the mean wavelength of light and L is the distance between the pinholes. With larger angles, the interference pattern washes out thus implying that the complex degree of coherence $\mu(\boldsymbol{r}_1, \boldsymbol{r}_2)$ is very small. Therefore, the distance $l_t \approx \bar{\lambda}/\theta_{\mathrm{s}}$ is called the transverse coherence length in the plane of screen and the coherence area at the corresponding plane must be given by [1, 4]

$$A_c \approx \left(\frac{\bar{\lambda}}{\theta_{\mathrm{s}}}\right)^2. \tag{3.10}$$

It should be emphasized that the cross–spectral density function does not represent the correlation of the Fourier transform of the random field $U(\boldsymbol{r}, t)$ but the correlation of random complex–amplitudes $V(\boldsymbol{r}, \omega)$ of the monochromatic field $V(\boldsymbol{r}, \omega)\exp(-i\omega t)$, despite the Fourier transform relation between cross spectral density $W(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega)$ and the mutual coherence function $\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2, \tau)$. Therefore, it can be written as [9, 10]

$$W(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega) = \langle V^*(\boldsymbol{r}_1, \omega) V(\boldsymbol{r}_2, \omega)\rangle. \tag{3.11}$$

For the special case of complete coherence in a volume, the correlation function can be expressed as its spatial factorization [4, 11]. In the space–time domain, it would mean if $|\gamma(\boldsymbol{r}_1, \boldsymbol{r}_2, \tau)| = 1$ for all τ and $\boldsymbol{r}_1, \boldsymbol{r}_2 \in D$ where D is some volume, then the mutual coherence function factors as

$$\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2, \tau) = V^*(\boldsymbol{r}_1) V(\boldsymbol{r}_2)\exp(-i\omega_0\tau), \tag{3.12}$$

where $V(\boldsymbol{r}) = \sqrt{I_1}\exp\left[-i\alpha(\boldsymbol{r})\right]$ is a position dependent function with $\alpha(\boldsymbol{r}) = \arg\{\gamma(\boldsymbol{r}_3, \boldsymbol{r}, \tau)\}$, $\boldsymbol{r}_3$ being a fixed point, and ω_0 is a constant. Likewise, in space frequency domain, complete coherence at a frequency ω in a certain volume assumes $|\mu(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega)| = 1$ and ensures the cross–spectral density function as [4]

$$W(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega) = F^*(\boldsymbol{r}_1, \omega) F(\boldsymbol{r}_2, \omega), \tag{3.13}$$

where $F(\boldsymbol{r},\omega) = \sqrt{S(\boldsymbol{r},\omega)}\exp\left[-i\beta(\boldsymbol{r},\omega)\right]$ is a function of the power density at $\boldsymbol{r}$ with $\beta(\boldsymbol{r},\omega) = \arg\{\mu(\boldsymbol{r}_1,\boldsymbol{r}_2,\omega)\}$. The function $F(\boldsymbol{r},\omega)$ satisfies the Helmholtz equation in free space and thus, can be treated as an electromagnetic field component. Therefore, a field coherent in a certain volume can be treated as a deterministic field, however, a field that is completely coherent at all frequencies for all points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ in certain volume does not necessitate the coherence of the field in general; the field is still random and may not be completely coherent in the space–time domain.

Chapter IV

Electromagnetic coherence theory

So far, we have assumed that the optical field has scalar nature, i.e., it is well directional and completely polarized in nature, which tremendously simplifies the characterization and analysis of the fields. However, the field is electromagnetic with the electric and magnetic components satisfying the Maxwell's equations [1] and propagating with a set of polarization properties and therefore, it is necessary to take on electromagnetic approach to fully understand its optical properties. At optical frequencies light–matter interaction does not involve magnetic fields, and hence it suffices to study properties of the electric field only. Furthermore, the study of partial coherence of general electromagnetic fields would be performed in the space–frequency domain, since it is a more convenient choice in optics due to its usefulness in analyzing broadband light.

Polarization is an important parameter of an optical field especially in laser, wireless and optical fibre telecommunications and radar. Polarization of light is a crucial parameter in several measurement techniques and has found ever-increasing applications in the field of engineering, geology, ellipsometry, and astronomy [12]. Some common applications involve polarized sunglasses, 3D glasses, radio transmission, or display technologies. Polarization is a property associated with waves that can oscillate in more than one direction. In optics, polarization of the field describes the direction in which the electric field oscillates with time [1, 2]. A perfectly polarized light, nonetheless, is an idealization of the real field. In practice, the polarization of the field at a point in space changes rapidly in a random manner, a consequence of superposition of polarized wavetrains generated randomly and independently from a large number of atomic emitters. Nevertheless, there is a certain degree of corre-

lation between the randomness in the polarization and hence, light whether natural or artificial is partially polarized in nature. [2] In this chapter, we study the properties of partially polarized light for 2D-fields before we proceed to examine the interference for electromagnetic fields in Young's two-pinhole experiment.

4.1 Electromagnetic cross-spectral density tensors

In the space–frequency domain, the coherence properties of a stationary electromagnetic field are described by correlation tensors [4]. Though we would be mainly focusing on the electric field, the correlation tensors discussed here are equally applicable to other vector fields as well. Let $E_i(\boldsymbol{r}, t)$ be any of the Cartesian components of the electric vector appearing in Maxwell's equations, then the mutual coherence tensor between the components is written as

$$\Gamma_{ij}(\boldsymbol{r}_1, \boldsymbol{r}_2, \tau) = \langle E_i^*(\boldsymbol{r}_1, t) E_j(\boldsymbol{r}_2, t+\tau) \rangle, \quad i = j = (x, y, z). \tag{4.1}$$

Also, the correlation–tensor functions follow the Hermiticity relation between the components:

$$[\Gamma_{ij}(\boldsymbol{r}_1, \boldsymbol{r}_2, \tau)^* = \Gamma_{ji}(\boldsymbol{r}_2, \boldsymbol{r}_1, \tau). \tag{4.2}$$

Analogously to the scalar approach, the electromagnetic cross–spectral density tensors $W_{ij}(\mathbf{r}_1, \mathbf{r}_2, \omega)$ can be expressed as the Fourier transforms of the correlation-tensor functions, where $\Gamma_{ij}(\boldsymbol{r}_1, \boldsymbol{r}_2, \tau)$ is assumed to be square integrable function. Therefore, we have

$$W_{ij}(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Gamma_{ij}(\boldsymbol{r}_1, \boldsymbol{r}_2, \tau) \exp(i\omega\tau)\, \mathrm{d}\tau, \tag{4.3}$$

whereas

$$\Gamma_{ij}(\boldsymbol{r}_1, \boldsymbol{r}_2, \tau) = \int_{0}^{\infty} W_{ij}(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega) \exp(-i\omega t)\, \mathrm{d}\omega. \tag{4.4}$$

It is easily seen that the cross spectral density tensor is a Hermitian tensor as well, i.e., $W_{ij}(\boldsymbol{r}_1, \boldsymbol{r}_2, \tau)^* = W_{ji}(\boldsymbol{r}_2, \boldsymbol{r}_1, \tau)$ and therefore it can be written in the matrix form as

$$\mathbf{W}^{\dagger}(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega) = \mathbf{W}(\boldsymbol{r}_2, \boldsymbol{r}_1, \omega), \tag{4.5}$$

where † denotes the conjugate transpose of the cross-spectral density tensor. It can also be deduced that the cross spectral density tensor can be understood as

the correlation between the vector complex amplitude $\mathbf{F}_{ij}(\boldsymbol{r},\omega)$ of an ensemble of monochromatic vector fields $\{\mathbf{F}(\boldsymbol{r},\omega)\exp(-i\omega t)\}$, analogous to the scalar fields [13, 14] and since $\mathbf{F}(\boldsymbol{r},\omega)$ also obeys the Helmholtz equation, it can be interpreted as an electric field component in space–frequency domain. Hence, the cross spectral density tensor can be written as

$$W_{ij}(\boldsymbol{r}_1,\boldsymbol{r}_2,\omega) = \langle F_i^*(\boldsymbol{r}_1,\omega)F_j(\boldsymbol{r}_2,\omega)\rangle \quad \text{or,} \tag{4.6}$$

$$\mathbf{W}(\boldsymbol{r}_1,\boldsymbol{r}_2,\omega) = \langle \mathbf{F}^*(\boldsymbol{r}_1,\omega)\mathbf{F}^{\mathrm{T}}(\boldsymbol{r}_2,\omega)\rangle. \tag{4.7}$$

4.2 Partial polarization

Any field at a point $\boldsymbol{r}$ and frequency ω is fully polarized if its realization $\mathbf{F}(\boldsymbol{r},\omega) = \alpha(\boldsymbol{r},\omega)\mathbf{V}(\boldsymbol{r},\omega)$ where $\alpha(\boldsymbol{r},\omega)$ is a complex random number and $\mathbf{V}(\boldsymbol{r},\omega)$ is a deterministic complex vector. On the contrary, an unpolarized field has no correlation between its components and the spectral densities in all directions are the same. The polarization property of a field in space–frequency domain is given by the second–order statistical entity, called the polarization matrix defined as [15, 16]

$$\mathbf{J}(\boldsymbol{r},\omega) = \mathbf{W}(\boldsymbol{r},\boldsymbol{r},\omega) = \langle \mathbf{F}^*(\boldsymbol{r},\omega)\mathbf{F}^{\mathrm{T}}(\boldsymbol{r},\omega)\rangle. \tag{4.8}$$

Like the cross–spectral density matrix, the polarization matrix is also Hermitian and non–negative definite [14]. If the field is well–directional, we may assume the propagation direction be one of the co–ordinate axis, supposedly $z-$axis, thereby resulting in a two–dimensional field. The polarization matrix of a two–dimensional field can hence be written as

$$\mathbf{J}(\boldsymbol{r},\omega) = \begin{bmatrix} J_{xx}(\boldsymbol{r},\omega) & J_{xy}(\boldsymbol{r},\omega) \\ J_{yx}(\boldsymbol{r},\omega) & J_{yy}(\boldsymbol{r},\omega) \end{bmatrix}, \tag{4.9}$$

where $J_{ij}(\boldsymbol{r},\omega) = \langle W_{ij}(\boldsymbol{r},\boldsymbol{r},\omega)\rangle$, $(i,j) = (x,y)$. If the field is fully polarized, the polarization matrix takes on the form

$$\mathbf{J}_{\mathrm{p}}(\boldsymbol{r},\omega) = \langle|\alpha(\boldsymbol{r},\omega)|^2\rangle \begin{bmatrix} |V_x(\boldsymbol{r},\omega)|^2 & V_x^*(\boldsymbol{r},\omega)V_y(\boldsymbol{r},\omega) \\ V_y^*(\boldsymbol{r},\omega)V_x(\boldsymbol{r},\omega) & |V_y(\boldsymbol{r},\omega)|^2 \end{bmatrix}, \tag{4.10}$$

where the subscript p stands for the polarized field. On the contrary, for unpolarized field, the correlation between the fields are defined as

$$\langle F_i(\boldsymbol{r},\omega)F_j(\boldsymbol{r},\omega)\rangle = \delta_{ij}A(\boldsymbol{r},\omega), \tag{4.11}$$

where $A(\boldsymbol{r},\omega) > 0$, and thus the polarization matrix from Eq. (4.9) results in

$$\mathbf{J}_{\mathrm{u}}(\boldsymbol{r},\omega) = A(\boldsymbol{r},\omega)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}. \tag{4.12}$$

As any random partially polarized field can be envisioned as a superposition of fully polarized and unpolarized fields, the polarization matrix of any arbitrary two-dimensional field can be broken into the factorized form:

$$\mathbf{J}(\boldsymbol{r},\omega) = \mathbf{J}_{\mathrm{p}}(\boldsymbol{r},\omega) + \mathbf{J}_{\mathrm{u}}(\boldsymbol{r},\omega). \tag{4.13}$$

The polarization state of the field can alternatively be defined in terms of Stokes parameter for two-dimensional fields as [4, 17]

$$\begin{aligned}
S_0(\boldsymbol{r},\omega) &= J_{xx}(\boldsymbol{r},\omega) + J_{yy}(\boldsymbol{r},\omega),\\
S_1(\boldsymbol{r},\omega) &= J_{xx}(\boldsymbol{r},\omega) - J_{yy}(\boldsymbol{r},\omega),\\
S_2(\boldsymbol{r},\omega) &= J_{yx}(\boldsymbol{r},\omega) + J_{xy}(\boldsymbol{r},\omega),\\
S_3(\boldsymbol{r},\omega) &= i[J_{yx}(\boldsymbol{r},\omega) - J_{xy}(\boldsymbol{r},\omega)],
\end{aligned} \tag{4.14}$$

where $S_j(\boldsymbol{r},\omega),\ j = 0\ldots3$ are purely real, the zeroth parameter representing the average spectral density of the field and others giving information about the polarization properties. Thus, the polarization matrix completely contains the information about the spectral density and the state of polarization [18]. The degree of polarization $P(\boldsymbol{r},\omega)$, on the other hand, is a measure of the polarized field in any arbitrary field, given by the ratio of the spectral density of the polarized light to the total spectral density [8, 16], i.e.,

$$P(\boldsymbol{r},\omega) = \frac{\mathrm{tr}\ \mathbf{J}_{\mathrm{p}}(\boldsymbol{r},\omega)}{\mathrm{tr}\ \mathbf{J}(\boldsymbol{r},\omega)} = \left[1 - 4\frac{\det\ \mathbf{J}(\boldsymbol{r},\omega)}{\mathrm{tr}^2\ \mathbf{J}(\boldsymbol{r},\omega)}\right]^{1/2} \tag{4.15}$$

where tr and det denote the trace and determinant of the matrix, respectively. Naturally, the degree of polarization has values in $0 \le P(\boldsymbol{r},\omega) \le 1$ where the values 0 or 1 stands for completely unpolarized or completely polarized fields, respectively.

4.3 Young's interference experiment

In the previous chapter, we studied the interference of scalar fields in view of Young's two-pinhole experiment. Now we consider the light to be partially polarized and the

polarization properties be modulated in the transverse direction, and we study their effects in the experiment. Further, the field is assumed to be well–directional which justifies a two–dimensional description of light. Following the same setup, illustrated in Figure 3.1, the field at any point $\boldsymbol{r}$ on the screen B for a frequency ω is expressed as [18, 19]

$$\boldsymbol{E}(\boldsymbol{r},\omega) = L_1 \boldsymbol{E}(\boldsymbol{S}_1,\omega)\frac{\exp(ikr_1)}{r_1} + L_2 \boldsymbol{E}(\boldsymbol{S}_2,\omega)\frac{\exp(ikr_2)}{r_2} \tag{4.16}$$

where $\boldsymbol{E}(\boldsymbol{S}_1,\omega)$ and $\boldsymbol{E}(\boldsymbol{S}_2,\omega)$ are the realizations of the fields at $\mathbf{S}_1$ and $\mathbf{S}_2$ respectively, k is the wavenumber, r_1 and r_2 hold the same meaning as in the scalar case, and L_1 and L_2 are purely imaginary numbers that depends on the area of the pinholes. The polarization matrix at the observation screen $\mathbf{J}(\boldsymbol{r},\omega)$, as defined in the previous section, can be derived from Eq. (4.16), resulting in [19]

$$\begin{aligned}\mathbf{J}(\boldsymbol{r},\omega) = \mathbf{J}^{(1)}(\boldsymbol{r},\omega) + \mathbf{J}^{(2)}(\boldsymbol{r},\omega) + \sqrt{S_0^{(1)}(\boldsymbol{r},\omega)S_0^{(2)}(\boldsymbol{r},\omega)}\\ \times\left\{\boldsymbol{\mu}(\boldsymbol{S}_1,\boldsymbol{S}_2,\omega)\exp[ik(R_2-R_1)] + \boldsymbol{\mu}(\boldsymbol{S}_2,\boldsymbol{S}_1,\omega)\exp[ik(R_1-R_2)]\right\},\end{aligned} \tag{4.17}$$

where $\mathbf{J}^{(\mathrm{j})}(\boldsymbol{r},\omega)$ and $S_0^{(j)}(\boldsymbol{r},\omega)$, $j=(1,2)$ are the polarization matrix and the zeroth Stokes parameter respectively at the screen B, under the case when only pinhole at S_j is open and

$$\boldsymbol{\mu}(\boldsymbol{S}_1,\boldsymbol{S}_2,\omega) = \frac{\mathbf{W}(\boldsymbol{S}_1,\boldsymbol{S}_2,\omega)}{\sqrt{S_0(\boldsymbol{S}_1,\omega)S_0(\boldsymbol{S}_2,\omega)}}, \tag{4.18}$$

is the normalized cross–spectral density matrix whose elements characterize the field correlations at the pinholes. To define the degree of coherence in the electromagnetic domain, one cannot simply extend the concept of complex degree of coherence from the scalar theory of partial coherence since the latter approach was essentially based on the scalar description of light. Karczewski [20, 21] and Wolf [22] defined the degree of coherence for electromagnetic fields in the space–time and space–frequency domains as the visibility of the interference fringes in Young's experiment, equivalently to the scalar case. From Eq. (4.17), we can write the spectral density at the

screen as

$$S_0(\boldsymbol{r},\omega) = S_0^{(1)}(\boldsymbol{r},\omega) + S_0^{(2)}(\boldsymbol{r},\omega) + 2\sqrt{S_0^{(1)}(\boldsymbol{r},\omega)S_0^{(2)}(\boldsymbol{r},\omega)}|\eta(\boldsymbol{S}_1,\boldsymbol{S}_2,\omega)|\cos[ik(R_2-R_1)+i\alpha(\boldsymbol{S}_1,\boldsymbol{S}_2,\omega)] \tag{4.19}$$

where $\eta(\boldsymbol{S}_1,\boldsymbol{S}_2,\omega) = \mathrm{tr}[\boldsymbol{\mu}(\boldsymbol{S}_1,\boldsymbol{S}_2,\omega)]$ is the complex degree of coherence, as suggested by Wolf and $\alpha(\boldsymbol{S}_1,\boldsymbol{S}_2,\omega) = \arg[\eta(\boldsymbol{S}_1,\boldsymbol{S}_2,\omega)]$ is its phase. This definition of degree of coherence is flawed, considering the fact that it bears no relation to the correlation between the fields, and does not remain invariant upon co–ordinate transformations. Therefore, alternative definitions of measure of coherence were suggested by Tervo, Setälä and Friberg as [14, 23]

$$\mu_{\mathrm{EM}}(\boldsymbol{S}_1,\boldsymbol{S}_2,\omega) = \|\boldsymbol{\mu}(\boldsymbol{S}_1,\boldsymbol{S}_2,\omega)\|_{\mathrm{F}}, \tag{4.20}$$

where $\|.\|_{\mathrm{F}}$ is the Euclidian norm. It is a real quantity having its value between 0 and 1, where 0 implies no correlations between the any field components at position $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ and 1 gives complete correlation. This definition of the degree of coherence for the electromagnetic fields remains invariant in unitary transformations, reduces to the magnitude of spectral degree of coherence under scalar case, i.e., $|\mu(\boldsymbol{S}_1,\boldsymbol{S}_2,\omega)|$ [23] and is consistent with Glauber's definition of complete coherence [11]. The degree of coherence relates back to the 2D–degree of polarization for identical values of fields, i.e., $\mathbf{E}(\boldsymbol{r}_1,\omega) = \mathbf{E}(\boldsymbol{r}_2,\omega)$ as [23]

$$\mu^2_{\mathrm{EM}}(\boldsymbol{r}_1,\boldsymbol{r}_2,\omega) = \frac{1}{2} + \frac{1}{2}P^2(\boldsymbol{r}_1,\omega), \tag{4.21}$$

where P is the 2D–degree of polarization as defined in Section 4.2. Equation (4.21) reveals that degree of coherence has dependence on the degree of polarization and may not be unity even for identical values of fields and specifically, the self–coherence of the fields is not satisfied; these properties has triggered an intense discussions for its validity [24–27].

A complete description of the spectral density as well as the polarization states of the resultant field at the screen is given by the so-called electromagnetic spectral

interference law [18, 28]:

$$S_j(\boldsymbol{r},\omega) = S_j^{(1)}(\boldsymbol{r},\omega) + S_j^{(2)}(\boldsymbol{r},\omega) + 2\sqrt{S_0^{(1)}(\boldsymbol{r},\omega)S_0^{(2)}(\boldsymbol{r},\omega)}|\eta_j(\boldsymbol{S}_1,\boldsymbol{S}_2,\omega)|\cos[ik(R_2-R_1)+i\alpha_j(\boldsymbol{S}_1,\boldsymbol{S}_2,\omega)], \tag{4.22}$$

where $S_j(\boldsymbol{r},\omega), j = 0, ...3$ are the classic Stokes parameters at point $\boldsymbol{r}$ at frequency ω and the superscripts (1) and (2) hold the same meaning as in Eq. (4.17) and $\eta_j(\boldsymbol{S}_1,\boldsymbol{S}_2,\omega)$ are the normalized two-point Stokes parameters defined as [18, 29, 30]:

$$\begin{aligned}
\eta_0(\boldsymbol{S}_1,\boldsymbol{S}_2,\omega) &= [W_{xx}(\boldsymbol{S}_1,\boldsymbol{S}_2,\omega) + W_{yy}(\boldsymbol{S}_1,\boldsymbol{S}_2,\omega)]/[S_0(\boldsymbol{S}_1,\omega)S_0(\boldsymbol{S}_2,\omega)]^{1/2},\\
\eta_1(\boldsymbol{S}_1,\boldsymbol{S}_2,\omega) &= [W_{xx}(\boldsymbol{S}_1,\boldsymbol{S}_2,\omega) - W_{yy}(\boldsymbol{S}_1,\boldsymbol{S}_2,\omega)]/[S_0(\boldsymbol{S}_1,\omega)S_0(\boldsymbol{S}_2,\omega)]^{1/2},\\
\eta_2(\boldsymbol{S}_1,\boldsymbol{S}_2,\omega) &= [W_{yx}(\boldsymbol{S}_1,\boldsymbol{S}_2,\omega) + W_{xy}(\boldsymbol{S}_1,\boldsymbol{S}_2,\omega)]/[S_0(\boldsymbol{S}_1,\omega)S_0(\boldsymbol{S}_2,\omega)]^{1/2},\\
\eta_3(\boldsymbol{S}_1,\boldsymbol{S}_2,\omega) &= i[W_{yx}(\boldsymbol{S}_1,\boldsymbol{S}_2,\omega) - W_{xy}(\boldsymbol{S}_1,\boldsymbol{S}_2,\omega)]/[S_0(\boldsymbol{S}_1,\omega)S_0(\boldsymbol{S}_2,\omega)]^{1/2}.
\end{aligned} \tag{4.23}$$

and $\alpha_j(\boldsymbol{S}_1,\boldsymbol{S}_2,\omega) = \arg[\eta_j(\boldsymbol{S}_1,\boldsymbol{S}_2,\omega)]$. Eq. (4.22) suggests that the interference in electromagnetic field includes not only the modulation of the intensities but also the modulation of the polarization properties, represented by the zeroth and the higher order Stokes parameter, respectively the latter being more important, at times, than the intensity itself [31]. Since the screen B is located far away from screen A, $S_j^{(1)}(\boldsymbol{r},\omega)$ and $S_j^{(2)}(\boldsymbol{r},\omega)$ vary very slowly with $\boldsymbol{r}$ and thus can be assumed as constants; consequently $S_j(\boldsymbol{r},\omega)$ is modulated sinusoidally in a transverse fashion due to the term $k(R_2 - R_1)$ [18]. The contrast of modulation (or visibilities) for Stokes parameters on the screen B, defined as

$$C_j = \frac{S_j(\boldsymbol{r},\omega)_{\max} - S_j(\boldsymbol{r},\omega)_{\min}}{S_0(\boldsymbol{r},\omega)_{\max} - S_0(\boldsymbol{r},\omega)_{\min}} \tag{4.24}$$

is related to the normalized Stokes parameter $|\eta_j(\boldsymbol{S}_1,\boldsymbol{S}_2,\omega)|$ and has its maximum value when the spectral densities at the screen are equal $S_0^{(1)}(\boldsymbol{r},\omega) = S_0^{(2)}(\boldsymbol{r},\omega)$:

$$C_j = |\eta_j(\boldsymbol{S}_1,\boldsymbol{S}_2,\omega)|. \tag{4.25}$$

This suggests that the contrast of modulation for the Stokes parameters on the screen is directly related to the correlation of the field components at the pinholes [18]. Under these circumstances, the electromagnetic degree of coherence and the

degree of polarization of the field at the pinholes can very well be determined from the modulation contrasts of the Stokes parameters. Therefore, we have [28, 32, 33]

$$\mu^2_{\mathrm{EM}}(\boldsymbol{S}_1, \boldsymbol{S}_2, \omega) = \frac{1}{2}\sum_{j=0}^{3} |\eta_j(\boldsymbol{S}_1, \boldsymbol{S}_2, \omega)|^2 = \frac{1}{2}\sum_{j=0}^{3} C_j^2, \tag{4.26}$$

$$P^2(\boldsymbol{r}_1, \omega) = \sum_{j=0}^{3} {s_j}^2 = \sum_{j=0}^{3} C_j^2 \tag{4.27}$$

where $s_j, j = 0 \ldots 3$ are the normalized Stokes parameters. Equation (4.26) implies that the electromagnetic degree of coherence can be physically interpreted as a direct measure of the contrasts of modulation of Stokes parameter, analogous to the scalar coherence whereas Eq. (4.27) shows that the degree of polarization of the field can be determined from the modulation contrasts when the beam interferes with itself.

Other propositions for the suitable measure of electromagnetic coherence include the work by Réfrégier and Goudail in 2005 [34, 35] and Luis in 2007 [36, 37].

Chapter V

Quantum-field theory of coherence

In this chapter, we discuss the quantum theory of coherence for scalar fields, i.e., in quantum–mechanical sense, the photons are polarized along a particular direction. Beginning with field correlations and quantum-mechanical first-order coherence functions, we finally give a quantum-mechanical description of Young's interference experiment. Therefore, this approach does not take into account the polarization properties of a full electromagnetic field, its relation with the correlation functions or its effect on Young's interference experiment. A full general treatment of photon polarization shall be discussed in Chapter 6.

5.1 Quantum optics

One of the most dominant and most researched fields of physics at present, quantum optics focusses on the light properties and its interaction with matter. With the discovery of light quanta, photons, several works were laid out by Schrödinger, Heisenberg, Bohr and Dirac that formed the foundations of quantum mechanics. Interest in quantum optics rose with more emphasis on the theory of photon statistics and photon counting. The first quantum description of interference was presented by Dirac [3], who explained the intensity pattern as a consequence of interference between the probability amplitudes of a photon to travel in either of the two paths and also concluded that a photon interferes only with itself, that conforms with the interference pattern emerging from one–photon interference experiment. Following the work of Dirac in quantum theory, Glauber, Wolf, Mandel, and many others contributed to the development of quantum theory of coherence. There are remarkable concepts of quantum optics such as quantum entanglement, that are actively

researched upon to realize quantum teleportation, quantum cryptography, quantum computation, long–distance quantum communication, etc [38].

5.2 Elements of the field theory

In quantum optics, we mainly work with observable quantities of the electromagnetic field, such as momentum, electric or magnetic field, among others. Throughout this paper, we will be focussing specifically on the electric field, which is represented by a Hermitian operator $\hat{\mathbf{E}}(\boldsymbol{r},t)$ upon quantization of the field. Furthermore, the electric field operator and the magnetic field operator satisfies Maxwell's equations [39]. The electric field operator can be decomposed into positive and negative frequency parts, i.e.,

$$\hat{\mathbf{E}}(\boldsymbol{r},t) = \hat{\mathbf{E}}^{(+)}(\boldsymbol{r},t) + \hat{\mathbf{E}}^{(-)}(\boldsymbol{r},t), \tag{5.1}$$

where $\hat{\mathbf{E}}^{(+)}(\boldsymbol{r},t) = [\hat{\mathbf{E}}^{(-)}(\boldsymbol{r},t)]^{\dagger}$. For a multimode field, the positive and negative frequency parts may be expanded as a superposition of modes and thus [40]

$$\hat{\mathbf{E}}^{(+)}(\boldsymbol{r},t) = i\sum_{\mathbf{k}s}\left(\frac{\hbar\omega_k}{2\varepsilon_0 V}\right)^{1/2}\hat{a}_{\mathbf{k}s}u_{\mathbf{k}s}(\boldsymbol{r})\exp\left(i\omega_k t\right), \tag{5.2}$$

$$\hat{\mathbf{E}}^{(-)}(\boldsymbol{r},t) = -i\sum_{\mathbf{k}s}\left(\frac{\hbar\omega_k}{2\varepsilon_0 V}\right)^{1/2}\hat{a}^{\dagger}_{\mathbf{k}s}u_{\mathbf{k}s}(\boldsymbol{r})\exp\left(-i\omega_k t\right), \tag{5.3}$$

where $\mathbf{k}$ is the wave vector, $s = (1,2)$ are two orthogonal polarizations, $\mathbf{k}s$ is the normal mode of the field, $u_{\mathbf{k}s}(\boldsymbol{r})$ is the mode function, and $\hat{a}_{\mathbf{k}s}$ and $\hat{a}^{\dagger}_{\mathbf{k}s}$ are the annihilation operator and the creation operator in the mode $\mathbf{k}s$ respectively. As it is evident from equations, each component is intrinsically complex, where the positive frequency part $\hat{\mathbf{E}}^{(+)}(\boldsymbol{r},t)$ is essentially a collective annihilation operator and hence associated with photon absorption, whereas its adjoint $\hat{\mathbf{E}}^{(-)}(\boldsymbol{r},t)$ with photon emission.

The annihilation operator $\hat{a}_{\mathbf{k}s}$ and the creation operator $\hat{a}^{\dagger}_{\mathbf{k}s}$ follow the boson commutation relations,

$$[\hat{a}_{\mathbf{k}s}, \hat{a}_{\mathbf{k}'s'}] = 0 = [\hat{a}^{\dagger}_{\mathbf{k}s}, \hat{a}^{\dagger}_{\mathbf{k}'s'}], \tag{5.4}$$

$$[\hat{a}_{\mathbf{k}s}, \hat{a}^{\dagger}_{\mathbf{k}'s'}] = \delta_{\mathbf{k}\mathbf{k}'}\delta_{\mathrm{ss}'}. \tag{5.5}$$

The number operator defined as $\hat{n}_{\mathbf{k}s} = \hat{a}^{\dagger}_{\mathbf{k}s}\hat{a}_{\mathbf{k}s}$ operates on a state as

$$\hat{n}_{\mathbf{k}s}|n_{\mathbf{k}s}\rangle = n_{\mathbf{k}s}|n_{\mathbf{k}s}\rangle, \tag{5.6}$$

where n_{ks} is the number of photons in $|n_{\mathbf{k}s}\rangle$. The multimode number state of the field is simply the product of the number states of all modes and may be written as [40]

$$|\{n_j\}\rangle = |n_1, n_2, n_3, \dots\rangle = \prod_{j=1}^{\infty} |n_j\rangle, \tag{5.7}$$

where $j = \mathbf{k}_j s_j$ denotes the normal mode of the field. These number states follow the orthonormality relation, i.e.,

$$\langle n_1, n_2, n_3, \dots | n_1', n_2', n_3', \dots\rangle = \delta_{n1n'1}\delta_{n2n'2}, \dots \tag{5.8}$$

and interacts with the annihilation operator and creation operator as follows:

$$\hat{a}_j|n_j\rangle = \sqrt{n_j}\ |n_j - 1\rangle, \tag{5.9}$$

$$\hat{a}_j^{\dagger}|n_j\rangle = \sqrt{n_j + 1}\ |n_j + 1\rangle. \tag{5.10}$$

These number states satisfy the Schrödinger equation such that

$$\hat{H}|\{n_j\}\rangle = E_n|\{n_j\}\rangle, \tag{5.11}$$

where $\hat{H}$ and E_n denotes the Hamiltonian operator and the energy eigenvalue of the field at state n, given by

$$\hat{H} = \sum_{\mathbf{k}s} \hbar\omega_k \left(\hat{a}_{\mathbf{k}s}^{\dagger}\hat{a}_{\mathbf{k}s} + \frac{1}{2}\right), \tag{5.12}$$

$$E_n = \sum_{\mathbf{k}s} \hbar\omega_k \left(n_{\mathbf{k}s} + \frac{1}{2}\right). \tag{5.13}$$

The electromagnetic field may be expressed in terms of coherent states, which form the eigenstates of the annihilation operator. Thus, we have [38]

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \tag{5.14}$$

where α is a complex number. The coherent state may be generated from the vacuum state when operated with displacement operator $\hat{D}(\alpha)$ and could be expanded in terms of the number states as

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle, \tag{5.15}$$

$$|\alpha\rangle = \exp\left[-(1/2)|\alpha|^2\right] \sum_{n=0}^{\infty} \frac{\alpha^n}{(n!)^{1/2}}|n\rangle. \tag{5.16}$$

It may be worth emphasizing that the quantum–mechanical analog of the detection process usually differs from its classical counterpart. In discussions of classical theory, one experimentally measures the classical field strength $\mathbf{E}(\boldsymbol{r}, t)$ that involves summing up the absorption and emission process. On the contrary, the quantum detection involves absorption of photons through photoionization or other means and therefore, only the positive frequency complex part plays a role in the coupling of the field to the matter [39]. If a field makes a transition from $|i\rangle$ to $|f\rangle$ upon absorption of photons, then the matrix element for the transition is given by [38,39]

$$\langle f|\hat{\mathbf{E}}^{(+)}(\boldsymbol{r}, t)|i\rangle, \tag{5.17}$$

and the probability rate for photons to be absorbed at point $\mathbf{r}$ and time t for an ideal photodetector is proportional to [39]

$$\sum_f |\langle f|\hat{\mathbf{E}}^{(+)}(\boldsymbol{r}, t)|i\rangle|^2 = \langle i|\hat{\mathbf{E}}^{(-)}(\boldsymbol{r}, t) \cdot \hat{\mathbf{E}}^{(+)}(\boldsymbol{r}, t)|i\rangle. \tag{5.18}$$

This gives the average counting rate of the detector for a field initially in a pure state. Nevertheless in practice, a field is more likely to be in mixed state and hence is described as an average over the ensemble states $\{|\psi_i\rangle\}$, given by the density operator

$$\hat{\rho} = \sum_i p_i |\psi_i\rangle\langle\psi_i|, \tag{5.19}$$

where p_i is the probability of the field being in ith state of the ensemble $\{|\psi_i\rangle\}$ and $\sum_i p_i = 1$. From the definition, it can be seen that the operator is Hermitian.

The expectation value of an operator $\hat{O}$ for a mixed state is expressed as

$$\langle\hat{O}\rangle = \mathrm{tr}(\hat{\rho}\hat{O}), \tag{5.20}$$

where tr stands for the trace of the matrix. Therefore, the average counting rate Eq. (5.18) can be rewritten as $\mathrm{tr}\{\hat{\rho}\hat{\mathbf{E}}^{(-)}(\boldsymbol{r}, t) \cdot \hat{\mathbf{E}}^{(+)}(\boldsymbol{r}, t)\}$.

5.3 Field correlations

A quantum theory of coherence can be formulated based on the observables, analogously to the classical theory. A quantum theory of photon detection shows that the intensity of a light beam is determined by measuring responses of the detecting

system that react by absorbing photons. Upon absorption of radiation, the states of both the atom and the field change and therefore the field intensity measured by an ideal photon detector at point $\boldsymbol{r}$ and time t can be described in terms of the transition probability of the field, resulting in [38, 40]

$$I(\boldsymbol{r},t) = \mathrm{tr}\Big\{\hat{\rho}\hat{\mathbf{E}}^{(-)}(\boldsymbol{r},t)\cdot\hat{\mathbf{E}}^{(+)}(\boldsymbol{r},t)\Big\}, \tag{5.21}$$

where $\hat{\rho}$ is the density operator describing the state of the field and the dot product denotes the scalar product. Equation (5.21) is a special form of a general function of considerable interest. When evaluated at any two arbitrary space–time points, the function furnishes a measure of the correlation between the respective fields, defined as the first–order correlation function

$$\mathrm{G}^{(1)}(x_1,x_2) = \mathrm{tr}\Big\{\hat{\rho}\hat{\mathbf{E}}^{(-)}(x_1)\cdot\hat{\mathbf{E}}^{(+)}(x_2)\Big\}, \tag{5.22}$$

where $x_i = (\boldsymbol{r}_i, t_i), i = (1,2)$. In the same manner, nth-order correlation function can be defined as [39]

$$\mathrm{G}^{(n)}(x_1,\dots x_n,x_{n+1},\dots x_{2n}) = \mathrm{tr}\Big\{\hat{\rho}\hat{\mathbf{E}}^{(-)}(x_1),\dots\hat{\mathbf{E}}^{(-)}(x_n)\cdot\hat{\mathbf{E}}^{(+)}(x_{n+1}),\dots\hat{\mathbf{E}}^{(+)}(x_{2n})\Big\}. \tag{5.23}$$

Analogously to the classical theory, the first-order correlation function can be normalized to yield the normalized first-order quantum coherence function defined as

$$g^{(1)}(x_1,x_2) = \frac{\mathrm{G}^{(1)}(x_1,x_2)}{[\mathrm{G}^{(1)}(x_1,x_1)\mathrm{G}^{(1)}(x_2,x_2)]^{1/2}}, \tag{5.24}$$

where $0 \leq |g^{(1)}(x_1,x_2)| \leq 1$. At this point, it seems justified to re-establish the concept of coherence and the necessary conditions, as we shall see shortly after the fields we have described previously as coherent do not even approximately obey the requirements. A fully coherent field has the normalized correlation functions all satisfying

$$|g^{(n)}(x_1,\dots x_{2n})| = 1, \quad n = 1,2,\dots \tag{5.25}$$

where the normalized correlation functions are defined as

$$g^{(n)}(x_1,\dots x_{2n}) = \frac{\mathrm{G}^{(n)}(x_1,\dots x_{2n})}{\prod_{j=1}^{2n}[\mathrm{G}^{(1)}(x_j,x_j)]^{1/2}}. \tag{5.26}$$

Thus, a field to be nth-order coherent requires $|g^{(j)}| = 1$ for $j \leq n$. In the discussions up to date about the optical studies, we have linked coherence to the first–order coherence and therefore, a classical field may be assumed to be first–order coherent, $|g^{(1)}| = 1$ but fails to show higher order coherence. A field is completely first–order coherent, given the numerator could be factorized as [38]

$$\mathrm{G}^{(1)}(x_1, x_2) = \left\langle \hat{\mathbf{E}}^{(-)}(x_1) \cdot \hat{\mathbf{E}}^{(+)}(x_2) \right\rangle = \left\langle \hat{\mathbf{E}}^{(-)}(x_1) \right\rangle \cdot \left\langle \hat{\mathbf{E}}^{(+)}(x_2) \right\rangle . \tag{5.27}$$

5.4 Young's interference experiment

Now we proceed to formulate the quantum description of the interference in Young's experiment. The experimental setup follows the same outline as presented earlier, and we assume that the fields are uniformly polarized and hence, may as well be treated as scalar fields. Following the same approach as in the classical scalar theory, we can rewrite the positive-frequency part of the electric field operator at any point $\boldsymbol{r}$ on the screen as the superposition of the corresponding parts of the fields from the two slits, i.e.,

$$\hat{E}^{(+)}(\boldsymbol{r}, t) = K_1 \hat{E}^{(+)}(\boldsymbol{S}_1, t_1) + K_2 \hat{E}^{(+)}(\boldsymbol{S}_2, t_2), \tag{5.28}$$

where, given $i = (1, 2)$, K_i is an imaginary number, $\hat{E}^{(+)}(\boldsymbol{S}_i, t_i)$ is the respective field operator at the slit at $\boldsymbol{S}_i$ and $t_i = t - r_i/c$, r_i being the distance from the slit to the point $\boldsymbol{r}$ on the screen. From Eq. (5.21), upon substitution of the electric field operators, the intensity becomes [38]

$$\begin{aligned} I(\boldsymbol{r}, t) &= |K_1|^2 G^{(1)}(x_1, x_1) + |K_2|^2 G^{(1)}(x_2, x_2) \\ &\quad + 2\sqrt{|K_1|^2 |K_2|^2 G^{(1)}(x_1, x_1) G^{(1)}(x_2, x_2)} \mathrm{Re}\{g^{(1)}(x_1, x_2)\}, \\ &= I_1 + I_2 + 2\sqrt{I_1 I_2} |g^{(1)}(x_1, x_2)| \cos\beta, \end{aligned} \tag{5.29}$$

where $x_i = (\boldsymbol{S}_i, t_i)$ and $\beta = \arg\{g^{(1)}(x_1, x_2)\}$. The first two terms describe the intensities on the screen when only slit at S_1 or S_2 is open whereas the last term gives the quantum interference term. Comparing with the classical interference law for partially incoherent fields, Eq. (3.3), we find a striking resemblance between them where the complex degree of coherence is replaced with the normalized first–order quantum coherence function, whose properties were discussed in the previous section. It also confirms that the intensity at the screen is critically affected by

the correlation of the fields at the slits, an intuitive notion that conforms with the classical theory.

Now we take on a special case where the incoming field is assumed to be monochromatic. With the dimensions of the pinholes of the order of wavelength of light, the pinholes act as secondary sources of spherical radiations. The mode function $u_{\mathbf{k}}(\mathbf{r})$ for spherical radiation takes the form [40]

$$u_{\mathbf{k}}(\mathbf{r}) = \mathbf{e}_{\mathbf{k}} \frac{\exp(ikr)}{r(4\pi R)^{1/2}}, \tag{5.30}$$

where $\mathbf{e}_{\mathbf{k}}$ is the unit polarization vector, R is the radius of the normalization volume, $r = |\mathbf{r}|$ and $k = |\mathbf{k}|$. Also, the unit polarization vector and the wave vector satisfies the transversality condition $\mathbf{k} \cdot \mathbf{e}_{\mathbf{k}} = 0$. As in the classical approach, the field at a point $\mathbf{r}$ and time t is the superposition of the spherical modes from the two pinholes. Substituting Eq. (5.30) into Eq. (5.2), we get

$$\hat{E}^{(+)}(\mathbf{r}, t) = f(\mathbf{r}, t)\big[\hat{a}_1 \exp(ikr_1) + \hat{a}_2 \exp(ikr_2)\big], \tag{5.31}$$

where $\hat{a}_1$ and $\hat{a}_2$ are the annihilation operators associated with the radial modes from pinholes 1 and 2 [38], r_1 and r_2 are the distances from the pinholes to the point $\mathbf{r}$ respectively and the funtion $f(\mathbf{r}, t)$ is given by

$$f(\mathbf{r}, t) = i \frac{(\hbar\omega)}{2\varepsilon_0}^{1/2} \frac{\mathbf{e}_{\mathbf{k}}}{r(4\pi R)^{1/2}} \exp(i\omega t), \tag{5.32}$$

where we have approximated $r_1 \approx r_2 \approx r$ in the denominators since the detecting screen is located far away from the pinholes. Substituting these values in Eq. (5.21), we have

$$\begin{aligned} I(\mathbf{r}, t) &= \mathrm{tr}\Big\{\hat{\rho}\hat{E}^{(-)}(\mathbf{r}, t)\hat{E}^{(+)}(\mathbf{r}, t)\Big\}, \\ &= |f(\mathbf{r}, t)|^2 \Big\{\mathrm{tr}\big(\hat{\rho}\hat{a}_1^\dagger \hat{a}_1\big) + \mathrm{tr}\big(\hat{\rho}\hat{a}_2^\dagger \hat{a}_2\big) + 2|\mathrm{tr}\big(\hat{\rho}\hat{a}_1^\dagger \hat{a}_2\big)| \cos\Phi\Big\}, \end{aligned} \tag{5.33}$$

Using Eq. (5.24), we can write the interference equation in a similar way as in classical theory. Thus, we get

$$I(\mathbf{r}, t) = |f(\mathbf{r}, t)|^2 \Big\{\mathrm{tr}\big(\hat{\rho}\hat{a}_1^\dagger \hat{a}_1\big) + \mathrm{tr}\big(\hat{\rho}\hat{a}_2^\dagger \hat{a}_2\big) + 2\sqrt{\mathrm{tr}\big(\hat{\rho}\hat{a}_1^\dagger \hat{a}_1\big)\mathrm{tr}\big(\hat{\rho}\hat{a}_2^\dagger \hat{a}_2\big)}|g^{(1)}| \cos\Phi\Big\}, \tag{5.34}$$

where $g^{(1)} = |g^{(1)}| \exp(i\phi)$ is the normalized first–order coherence function, defined in Eq. (5.24), specifically between the modes $\hat{a}_1$ and $\hat{a}_2$ and $\Phi = k(r_1 - r_2) + \phi$. The values of $|g^{(1)}|$ is constrained within the limit $0 \leq |g^{(1)}| \leq 1$ where the extremes 0 or 1 correspond to incoherence or first–order coherence respectively [41]. It is obvious that the interference fringes will be visible as long as there exists some correlation between the modes, i.e., $|g^{(1)}| \neq 0$. As we move the observation point about the screen, the phase relation between the fields changes, causing maxima in some and minima in others. The maximum intensity occurs when $\Phi = 2\pi n$ where $n = 0, 1, 2, \ldots$. Furthermore, it could be noted that the maximum intensity reduces by a factor $1/r^2$ as it moves away from the central fringe [38, 40].

The fringe visibility, as defined earlier, takes the form [40]

$$V = \frac{2\sqrt{I_1 I_2}}{I_1 + I_2} |g^{(1)}|, \tag{5.35}$$

where $I_i = |f(\mathbf{r}, t)|^2 \mathrm{tr}\big(\hat{\rho}\hat{a}_i^\dagger \hat{a}_i\big), i = (1, 2)$, are the intensities of the radial modes $\hat{a}_i$. It could be realized that the contrast of the fringes will be maximum, $V = 1$, for fields having equal intensities for each mode and possessing first-order coherence, i.e., $|g^{(1)}| = 1$.

Next, we investigate the interference of some special fields, which are initially in a pure state $|\psi\rangle$. A pure state could be generated from a single-mode excitation of the vacuum states [40], and therefore we may write

$$|\psi\rangle = h(\hat{a}^\dagger)|0\rangle, \tag{5.36}$$

where h is any function and $\hat{a}^\dagger$ is the creation operator for a single mode of the radiation field. In Young's experiment, the incident field can be assumed as a single–mode plane wave with the annihilation operator $\hat{a}$ that may be expressed as a linear combination of operators of the radial modes $\hat{a}_1$ and $\hat{a}_2$, i.e.,

$$\hat{a} = \hat{a}_1 \cos\theta + \hat{a}_2 \sin\theta, \tag{5.37}$$

where θ is a measure of the amplitudes of the modes. If the pinholes are of equal size and given each slit has a detector behind it, then a photon has equal probability of passing through either of them and therefore, the single–mode operator becomes

$$\hat{a} = \frac{1}{\sqrt{2}}\big(\hat{a}_1 + \hat{a}_2\big), \tag{5.38}$$

where the operators of the modes are supposed to satisfy the boson commutation relations

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}, \quad [\hat{a}_i, \hat{a}_j] = 0 = [\hat{a}_i^\dagger, \hat{a}_j^\dagger], \quad [\hat{a}, \hat{a}^\dagger] = 1, \quad (i, j = 1, 2) \tag{5.39}$$

but upon substitution of the radial mode's operators, we see that the first relation can not be satisfied. Thus, we introduce a fictitious mode b which always exist in the vacuum state and whose operator is defined as $\hat{b} = (\hat{a}_1 - \hat{a}_2)/\sqrt{2}$ where $[\hat{b}, \hat{b}^\dagger] = 1$; this satisfies the unitary transformation between the input and the output values too. In general, $n-$photon state of mode a can be related to the vacuum state at modes 1 and 2 as [38]

$$|n\rangle_a|0\rangle_b = \frac{1}{\sqrt{n!}}\hat{a}^{\dagger n}|0\rangle_a|0\rangle_b = \frac{1}{\sqrt{n!}}\left(\frac{1}{\sqrt{2}}\right)^n(\hat{a}_1^\dagger + \hat{a}_2^\dagger)^n|0\rangle_1|0\rangle_2 \tag{5.40}$$

One–photon field can be decomposed in terms of the vacuum states of the radial modes as

$$|1\rangle_a|0\rangle_b = \left(\frac{1}{\sqrt{2}}\right)(|1,0\rangle + |0,1\rangle), \tag{5.41}$$

where the notation $|m, n\rangle$ means m photons in mode 1 and n photons in mode 2; this equation illustrates that a single photon incident at the screen has an equal probability of passing through each pinholes and the interference actually occurs between the probabilities of the photon to pass through different pinholes. Substituting this initial state in Eq. (5.34) and simplifying, we get [40]

$$I(\mathbf{r}, t) = |f(\mathbf{r}, t)|^2(1 + \cos\Phi), \tag{5.42}$$

Evidently, the intensity fringes develop on the screen with the succession of one–photon interference, which supports the theory that a photon interferes with itself. Furthermore, we see that one–photon field exhibits first–order coherence since $|g^{(1)}| = 1$. In the same manner, a two–photon field can be expressed as

$$|2\rangle_a|0\rangle_b = \frac{1}{2}\big(|2,0\rangle + \sqrt{2}|1,1\rangle + |0,2\rangle\big), \tag{5.43}$$

which results in the intensity as

$$I(\mathbf{r}, t) = 2|f(\mathbf{r}, t)|^2\big(1 + \cos\Phi\big), \tag{5.44}$$

and thus generalizing for a $n-$photon field gives [38]

$$I(\mathbf{r},t) = n|f(\mathbf{r},t)|^2(1+\cos\Phi) \tag{5.45}$$

If the incident field is in coherent state $|\alpha\rangle$, then using the displacement operator, defined previously and the operators relation, we get

$$|\alpha\rangle_a|0\rangle_b = \left|\frac{\alpha}{\sqrt{2}}\right\rangle_1 \left|\frac{\alpha}{\sqrt{2}}\right\rangle_2, \tag{5.46}$$

where each mode is in coherent states as well and gives the intensity pattern as

$$I(\mathbf{r},t) = |\alpha|^2|f(\mathbf{r},t)|^2\big(1+\cos\Phi\big), \tag{5.47}$$

The two interfering modes in all these cases are first–order coherent, except in case of coherent states where they possess coherence of all states, and produces the same interference pattern that agrees with the classical interference result for field with equal intensities at the pinholes. Nevertheless, the strength of the fringes depends on the photons incident on the screen and therefore, varies with the field. It is worth noting that the interference between independent light beams is only possible for certain states, for example coherent states where the modes in the coherent states be $|\alpha_1\rangle|\alpha_2\rangle$ can arise from independent laser beams. If the modes are independent and in Fock states, then the product number state $|n_1\rangle|n_2\rangle$ yields a zero correlation function and thus no interference fringes [38, 40].

Chapter VI

Quantum analysis of electromagnetic field

In the previous chapter, we investigated the quantum analysis of coherence of scalar fields, where the field was projected parallel along a single unit vector $\mathbf{e_k}$. To account for the full electromagnetic nature of radiation, the polarization property of the field must be addressed. In this chapter, we present the polarization of an arbitrary field and study its interference in Young's experiment from a quantum-mechanical point of view.

6.1 Polarization property of a field

The correlation functions for the components of the field is defined as [42]

$$\mathrm{G}^{(1)}_{ij}(x_1, x_2) = \mathrm{Tr}\big\{\hat{\rho}\hat{E}^{(-)}_i(x_1)\hat{E}^{(+)}_j(x_2)\big\}, \tag{6.1}$$

where $(i, j) = (x, y, z)$ are the Cartesian components of the vector field and $x_k = (\boldsymbol{r}_k, t_k)$, $k = (1, 2)$, a simple generalization of the scalar correlation defined previously. These functions satisfy the symmetry relation and obey the inequalities [39]

$$\mathrm{G}^{(1)}_{ij}(x_1, x_2) = \{\mathrm{G}^{(1)}_{ji}(x_2, x_1)\}^*, \tag{6.2}$$

$$\mathrm{G}^{(1)}_{ii}(x_1, x_1) \geq 0, \tag{6.3}$$

$$\mathrm{G}^{(1)}_{ii}(x_1, x_1)\mathrm{G}^{(1)}_{jj}(x_2, x_2) \geq |\mathrm{G}^{(1)}_{ij}(x_1, x_2)|. \tag{6.4}$$

The first-order correlation functions of the field, which correspond to the second-order in the classical theory, can be summarized in terms of the 3×3 correlation tensor $\mathbf{G}^{(1)}(x_1, x_2)$ (which takes 2×2 form for a beamlike field) such that

$$\mathbf{G}^{(1)}(x_1, x_2) = \{\mathrm{G}^{(1)}_{ij}(x_1, x_2)\} \tag{6.5}$$

A special kind of such matrix is the quantum polarization matrix $\mathbf{G}^{(1)}(x,x)$, defined as the correlation matrix for equal points in space–time, $x_1 = x_2 = x$, which contribute to photon counting rate. From Cauchy–Schwarz inequality, it follows that the quantum polarization matrix is non–negative definite and its elements for a polarized field can be written as [39, 42]

$$\mathrm{G}^{(1)}_{xx}(x,x) = I(x)_{\theta=0,\alpha=0}, \tag{6.6}$$

$$\mathrm{G}^{(1)}_{yy}(x,x) = I(x)_{\theta=\pi/2,\alpha=0}, \tag{6.7}$$

$$\mathrm{G}^{(1)}_{xy}(x,x) = \frac{1}{2}[I(x)_{\theta=\pi/4,\alpha=0} - I(x)_{\theta=3\pi/4,\alpha=0}] + \frac{i}{2}[I(x)_{\theta=\pi/4,\alpha=\pi/2} - I(x)_{\theta=3\pi/4,\alpha=\pi/2}], \tag{6.8}$$

$$\mathrm{G}^{(1)}_{yx}(x,x) = \frac{1}{2}[I(x)_{\theta=\pi/4,\alpha=0} - I(x)_{\theta=3\pi/4,\alpha=0}] - \frac{i}{2}[I(x)_{\theta=\pi/4,\alpha=\pi/2} - I(x)_{\theta=3\pi/4,\alpha=\pi/2}], \tag{6.9}$$

where $I(x)$ denotes the average photon counting rate at point x, θ stands for the polarization angle of the field with $x-$axis and $\alpha = \alpha_y - \alpha_x$ gives the phase difference between the x and y components. In the above formulation and henceforth, we will assume a field propagating along $z$axis and hence the tensor reduces to a 2×2 matrix. The photon counting rate of a photodetector can also be written as the trace of the quantum polarization matrix [42], analogous to the relation between intensity and polarization matrix in classical theory

$$I(x) = \mathrm{Tr}\{\mathbf{G}^{(1)}(x,x)\}. \tag{6.10}$$

It is convenient to use the normalized correlation function $g^{(1)}_{ij}(x)$, defined as

$$g^{(1)}_{ij}(x) = \frac{\mathrm{G}^{(1)}_{ij}(x,x)}{\sqrt{\mathrm{G}^{(1)}_{ii}(x,x)\mathrm{G}^{(1)}_{jj}(x,x)}} = |g^{(1)}_{ij}(x)| \exp\left[i\beta_{ij}(x)\right] \tag{6.11}$$

where it obeys $0 \leq |g^{(1)}_{xy}(x)| \leq 1$, for $(i,j) = (x,y)$. For an unpolarized beam, $|g^{(1)}_{xy}(x)| = 0$ and the quantum polarization matrix becomes proportional to the unit matrix, expressed as [42]

$$\mathbf{G}^{(1)}_{(\mathrm{u})}(x,x) = A(x) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{6.12}$$

where $A(x) = \mathrm{G}^{(1)}_{xx}(x,x) = \mathrm{G}^{(1)}_{yy}(x,x)$ is a real function of space and time and the subscript (u) denotes the unpolarized field. On the other hand, when the field is completely polarized, i.e., $|g^{(1)}_{xy}(x)| = 1$, then from Eq. (6.11), it can be deduced that the elements of the quantum polarization matrix be written in the factorized form as [42]

$$\{\mathbf{G}^{(1)}_{(\mathrm{p})}(x,x)\}_{ij} = F_i^*(x)F_j(x), \quad (i,j) = (x,y), \tag{6.13}$$

where $F_i(x) = [G^{(1)}_{ii}(x,x)]^{1/2}\exp\{i\phi_{ii}(x)\}$ where $\phi_{yy}(x) - \phi_{xx}(x) = \beta_{xy}(x)$ is the phase of $g^{(1)}_{xy}(x)$, as mentioned above. It follows from the equations above that for a completely polarized beam, the determinant of the quantum polarization matrix becomes zero, i.e., $\det[\mathbf{G}_{(p)}(x,x)] = 0$. Analogously to the classical concept, the quantum polarization matrix of a partially polarized field can be decomposed into the matrices of the polarized part and the unpolarized part of the field [42]

$$\mathbf{G}^{(1)}(x,x) = \mathbf{G}^{(1)}_{(\mathrm{p})}(x,x) + \mathbf{G}^{(1)}_{(\mathrm{u})}(x,x) \tag{6.14}$$

and as a consequence, the average photon counting rate of the photodetector from Eq. (6.10) follows a similar relation. With analogy to the classical theory, the degree of polarization can be defined as the ratio of the photon counting rate of the polarized part to the total counting rate, which upon simplification takes the form, [42]

$$P(x) = \frac{\mathrm{tr}\{\mathbf{G}^{(1)}_{(p)}(x,x)\}}{\mathrm{tr}\{\mathbf{G}^{(1)}(x,x)\}} = \sqrt{1 - \frac{4\ \det\{\mathbf{G}^{(1)}(x,x)\}}{\mathrm{tr}\{\mathbf{G}^{(1)}(x,x)\}}} \tag{6.15}$$

wherefrom one can see that $P(x) = 0$ for unpolarized light and $P(x) = 1$ for completely polarized light. It also occurs that the quantum degree of polarization bears peculiar resemblance to the classical degree of polarization, as discussed earlier.

6.2 Stokes parameters in quantum mechanics

We will build the Stokes parameters from the quantum polarization matrix, in a manner similar to classical theory. Thus, the quantum Stokes parameter are defined as

$$S_0(\boldsymbol{r},t) = \mathrm{G}^{(1)}_{xx}(\boldsymbol{r},t;\boldsymbol{r},t) + \mathrm{G}^{(1)}_{yy}(\boldsymbol{r},t;\boldsymbol{r},t), \tag{6.16a}$$

$$S_1(\boldsymbol{r},t) = \mathrm{G}^{(1)}_{xx}(\boldsymbol{r},t;\boldsymbol{r},t) - \mathrm{G}^{(1)}_{yy}(\boldsymbol{r},t;\boldsymbol{r},t), \tag{6.16b}$$

$$S_2(\boldsymbol{r},t) = \mathrm{G}^{(1)}_{yx}(\boldsymbol{r},t;\boldsymbol{r},t) + \mathrm{G}^{(1)}_{xy}(\boldsymbol{r},t;\boldsymbol{r},t), \tag{6.16c}$$

$$S_3(\boldsymbol{r},t) = i[\mathrm{G}^{(1)}_{yx}(\boldsymbol{r},t;\boldsymbol{r},t) + \mathrm{G}^{(1)}_{xy}(\boldsymbol{r},t;\boldsymbol{r},t)]. \tag{6.16d}$$

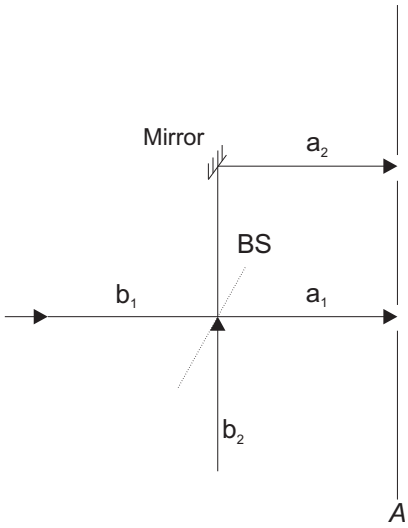

Figure 6.1: The quantum-mechanical depiction of input and output fields in Young's experiment.

Coming back to the interference experiment, let us consider a non-polarizing 50 : 50 beam splitter is placed ahead of the screen with the pinholes such that the first exit leads to the pinhole 1 and the second exit leads to the pinhole 2. The construction of the beam splitter determines the phase difference between the reflected and transmitted beams and we shall assume here that the reflected beam suffers a phase shift of $\pi/2$. Thefore, we introduce a $-\pi/2$ compensator at the second arm, as illustrated in Figure 6.1. In the figure, $\hat{b}_1$ represents the annihilation operator of the incident mode whereas $\hat{b}_2$ assumes the fictitious mode; the annihilation operators of the output radial modes are denoted by $\hat{a}_1$ and $\hat{a}_2$. The input-output relations of the beam splitter along with the compensator are given by

$$\hat{a}_1 = \frac{1}{\sqrt{2}}(\hat{b}_1 + i\hat{b}_2), \quad \hat{a}_2 = \frac{1}{\sqrt{2}}(\hat{b}_1 - i\hat{b}_2), \tag{6.17}$$

whereas the operators of the input modes are written as

$$\hat{b}_1 = \frac{1}{\sqrt{2}}(\hat{a}_1 + \hat{a}_2), \quad \hat{b}_2 = -\frac{i}{\sqrt{2}}(\hat{a}_1 - \hat{a}_2). \tag{6.18}$$

With electromagnetic approach, the operators can be decomposed into their Cartesian components, i.e., x and y components, each of which satisfies Eq. (6.17) and

Eq. (6.18). Therefore, we have

$$\hat{a}_{1x} = \frac{1}{\sqrt{2}}(\hat{b}_{1x} + i\hat{b}_{2x}), \quad \hat{a}_{2x} = \frac{1}{\sqrt{2}}(\hat{b}_{1x} - i\hat{b}_{2x}),$$
$$\hat{a}_{1y} = \frac{1}{\sqrt{2}}(\hat{b}_{1y} + i\hat{b}_{2y}), \quad \hat{a}_{2y} = \frac{1}{\sqrt{2}}(\hat{b}_{1y} - i\hat{b}_{2y}), \tag{6.19}$$

and

$$\hat{b}_{1x} = \frac{1}{\sqrt{2}}(\hat{a}_{1x} + \hat{a}_{2x}), \quad \hat{b}_{2x} = -\frac{i}{\sqrt{2}}(\hat{a}_{1x} - \hat{a}_{2x}),$$
$$\hat{b}_{1y} = \frac{1}{\sqrt{2}}(\hat{a}_{1y} + \hat{a}_{2y}), \quad \hat{b}_{2y} = -\frac{i}{\sqrt{2}}(\hat{a}_{1y} - \hat{a}_{2y}). \tag{6.20}$$

To investigate the Stokes parameters at the screen, we assume that the incident field is monochromatic and the transmitted field at any point on the screen as the superposition of the spherical modes from the pinholes, as we did in the quantum approach to scalar field. Using Eq. (5.31) and Eq. (6.1) into Eq. (6.16), the Stokes parameters take the forms

$$\begin{aligned}
S_0(\boldsymbol{r},t) &= |f(\boldsymbol{r},t)|^2\big[\mathrm{Tr}\big(\hat{\rho}\hat{a}_{1x}^\dagger\hat{a}_{1x}\big) + \mathrm{Tr}\big(\hat{\rho}\hat{a}_{2x}^\dagger\hat{a}_{2x}\big) + \mathrm{Tr}\big(\hat{\rho}\hat{a}_{1y}^\dagger\hat{a}_{1y}\big) + \mathrm{Tr}\big(\hat{\rho}\hat{a}_{2y}^\dagger\hat{a}_{2y}\big) \\
&\quad + 2|\mathrm{Tr}\big(\hat{\rho}\hat{a}_{1x}^\dagger\hat{a}_{2x}\big)|\cos\Phi_{xx} + 2|\mathrm{Tr}\big(\hat{\rho}\hat{a}_{1y}^\dagger\hat{a}_{2y}\big)|\cos\Phi_{yy}\big], \qquad (6.21a) \\
S_1(\boldsymbol{r},t) &= |f(\boldsymbol{r},t)|^2\big[\mathrm{Tr}\big(\hat{\rho}\hat{a}_{1x}^\dagger\hat{a}_{1x}\big) + \mathrm{Tr}\big(\hat{\rho}\hat{a}_{2x}^\dagger\hat{a}_{2x}\big) - \mathrm{Tr}\big(\hat{\rho}\hat{a}_{1y}^\dagger\hat{a}_{1y}\big) - \mathrm{Tr}\big(\hat{\rho}\hat{a}_{2y}^\dagger\hat{a}_{2y}\big) \\
&\quad + 2|\mathrm{Tr}\big(\hat{\rho}\hat{a}_{1x}^\dagger\hat{a}_{2x}\big)|\cos\Phi_{xx} - 2|\mathrm{Tr}\big(\hat{\rho}\hat{a}_{1y}^\dagger\hat{a}_{2y}\big)|\cos\Phi_{yy}\big], \qquad (6.21b) \\
S_2(\boldsymbol{r},t) &= 2|f(\boldsymbol{r},t)|^2\big\{\big[\mathrm{Tr}\big(\hat{\rho}\hat{a}_{1x}^\dagger\hat{a}_{1y}\big)\big] + \mathrm{Re}\big[\mathrm{Tr}\big(\hat{\rho}\hat{a}_{2x}^\dagger\hat{a}_{2y}\big)\big] \\
&\quad + |\mathrm{Tr}\big(\hat{\rho}\hat{a}_{1x}^\dagger\hat{a}_{2y}\big)|\cos\Phi_{xy} + |\mathrm{Tr}\big(\hat{\rho}\hat{a}_{1y}^\dagger\hat{a}_{2x}\big)|\cos\Phi_{yx}\big\}, \qquad (6.21c) \\
S_3(\boldsymbol{r},t) &= 2|f(\boldsymbol{r},t)|^2\big\{\big[\mathrm{Im}\big(\hat{\rho}\hat{a}_{1x}^\dagger\hat{a}_{1y}\big)\big] + \mathrm{Im}\big[\mathrm{Tr}\big(\hat{\rho}\hat{a}_{2x}^\dagger\hat{a}_{2y}\big)\big] \\
&\quad - |\mathrm{Tr}\big(\hat{\rho}\hat{a}_{1y}^\dagger\hat{a}_{2x}\big)|\sin\Phi_{yx} + |\mathrm{Tr}\big(\hat{\rho}\hat{a}_{1x}^\dagger\hat{a}_{2y}\big)|\sin\Phi_{xy}\big\}, \qquad (6.21d)
\end{aligned}$$

where Re and Im denotes the real and imaginary parts, respectively, and

$$\Phi_{ij} = k(r_2 - r_1) + \arg\big[\mathrm{Tr}\big(\hat{\rho}\hat{a}_{1i}^\dagger\hat{a}_{2j}\big)\big], \quad (i,j) = (x,y). \tag{6.22}$$

In order to obtain the explicit expressions for the Stokes parameters, the traces must be evaluated which requires us to define the state of the incident field. A one-photon field can be treated as a scalar field since the polarization of photon does not change; the polarization effects come into effect for higher-order photon field and therefore, we shall present a specific case of a two-photon field.

6.3 A two-photon field interference

We consider a general form of a pure two-photon field given in terms of entangled photon pairs

$$|\psi\rangle' = \alpha|2,0,0,0\rangle' + \beta|1,1,0,0\rangle' + \gamma|0,2,0,0\rangle', \tag{6.23}$$

where $|m\rangle_{1x}|n\rangle_{1y}|p\rangle_{2x}|q\rangle_{2y} = |m,n,p,q\rangle$ and the prime indicates the input state. The amplitudes of the states serves as a measure of their probabilities and obeys

$$|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1. \tag{6.24}$$

Substituting the relations from Eq . (6.20) into the Eq. (6.23), we find entangled output states as

$$\begin{aligned}|\psi\rangle' \xrightarrow{\text{BS}} &\frac{\alpha}{2}\big[|2,0,0,0\rangle + |0,0,2,0\rangle + \sqrt{2}|1,0,1,0\rangle\big] \\ &+ \frac{\beta}{2}\big[|1,1,0,0\rangle + |1,0,0,1\rangle + |0,1,1,0\rangle + |0,0,1,1\rangle\big] \\ &+ \frac{\gamma}{2}\big[|0,2,0,0\rangle + |0,0,0,2\rangle + \sqrt{2}|0,1,0,1\rangle\big].\end{aligned} \tag{6.25}$$

Operating on the state $|\varphi\rangle'$ defined in Eq. (6.25), we obtain the trace terms of Eq. (6.21) as

$$\mathrm{Tr}(\hat{\rho}\hat{a}_{1x}^\dagger\hat{a}_{1x}) = |\alpha|^2 + \frac{\beta^2}{2}, \qquad \mathrm{Tr}(\hat{\rho}\hat{a}_{1y}^\dagger\hat{a}_{1y}) = |\gamma|^2 + \frac{\beta^2}{2}, \tag{6.26a}$$

$$\mathrm{Tr}(\hat{\rho}\hat{a}_{2x}^\dagger\hat{a}_{2x}) = |\alpha|^2 + \frac{\beta^2}{2}, \qquad \mathrm{Tr}(\hat{\rho}\hat{a}_{2y}^\dagger\hat{a}_{2y}) = |\gamma|^2 + \frac{\beta^2}{2}, \tag{6.26b}$$

$$\mathrm{Tr}(\hat{\rho}\hat{a}_{1x}^\dagger\hat{a}_{1y}) = \frac{\beta}{\sqrt{2}}(\alpha^* + \gamma), \quad \mathrm{Tr}(\hat{\rho}\hat{a}_{1y}^\dagger\hat{a}_{1x}) = \frac{\beta}{\sqrt{2}}(\alpha + \gamma^*), \tag{6.26c}$$

$$\mathrm{Tr}(\hat{\rho}\hat{a}_{2x}^\dagger\hat{a}_{2y}) = \frac{\beta}{\sqrt{2}}(\alpha^* + \gamma), \quad \mathrm{Tr}(\hat{\rho}\hat{a}_{2y}^\dagger\hat{a}_{2x}) = \frac{\beta}{\sqrt{2}}(\alpha + \gamma^*), \tag{6.26d}$$

$$\mathrm{Tr}(\hat{\rho}\hat{a}_{1x}^\dagger\hat{a}_{2x}) = |\alpha|^2 + \frac{\beta^2}{2}, \qquad \mathrm{Tr}(\hat{\rho}\hat{a}_{1y}^\dagger\hat{a}_{2y}) = |\gamma|^2 + \frac{\beta^2}{2}, \tag{6.26e}$$

$$\mathrm{Tr}(\hat{\rho}\hat{a}_{1x}^\dagger\hat{a}_{2y}) = \frac{\beta}{\sqrt{2}}(\alpha^* + \gamma), \quad \mathrm{Tr}(\hat{\rho}\hat{a}_{1y}^\dagger\hat{a}_{2x}) = \frac{\beta}{\sqrt{2}}(\alpha + \gamma^*), \tag{6.26f}$$

where we have assumed that β is real. Substituting these in Eq. (6.21) and using relation Eq. (6.24), we get the Stokes parameter as

$$S_0(\boldsymbol{r},t) = 2|f(\boldsymbol{r},t)|^2(1+\cos\Phi), \tag{6.27a}$$
$$S_1(\boldsymbol{r},t) = 2|f(\boldsymbol{r},t)|^2(|\alpha|^2 - |\gamma|\ 2)(1+\cos\Phi), \tag{6.27b}$$
$$S_2(\boldsymbol{r},t) = 2\sqrt{2}|f(\boldsymbol{r},t)|^2\beta|\alpha^* + \gamma|\cos\xi(1+\cos\Phi), \tag{6.27c}$$
$$S_3(\boldsymbol{r},t) = 2\sqrt{2}|f(\boldsymbol{r},t)|^2\beta|\alpha^* + \gamma|\sin\xi(1+\cos\Phi), \tag{6.27d}$$

where $\Phi = k(r_2 - r_1)$ and $\xi = \arg(\alpha^* + \gamma)$. The first Stokes parameter $S_0(\boldsymbol{r},t)$ gives the intensity of the field at a given point and is parallel to the derivation in scalar case Eq. (5.44) whereas the higher Stokes parameters describe the polarization properties. The Stokes parameters of the input state $|\psi\rangle'$ can be found out by operating the state by the creation and annihilation operators of the incident mode $\hat{b}_1$. Therefore, we have

$$\hat{b}_x|\psi\rangle' = \alpha\sqrt{2}|1,0\rangle' + \beta|0,1\rangle', \tag{6.28}$$
$$\hat{b}_y|\psi\rangle' = \beta\sqrt{2}|1,0\rangle' + \gamma\sqrt{2}|0,1\rangle'. \tag{6.29}$$

Employing these values in Eq. (6.16), we get

$$S_0' = \langle\psi|\hat{b}_x^\dagger\hat{b}_x|\rangle' + \langle\psi|\hat{b}_y^\dagger\hat{b}_y|\rangle' = 2, \tag{6.30a}$$
$$S_1' = \langle\psi|\hat{b}_x^\dagger\hat{b}_x|\rangle' - \langle\psi|\hat{b}_y^\dagger\hat{b}_y|\rangle' = 2(|\alpha|^2 - |\gamma|^2), \tag{6.30b}$$
$$S_2' = \langle\psi|\hat{b}_x^\dagger\hat{b}_y|\rangle' - \langle\psi|\hat{b}_y^\dagger\hat{b}_x|\rangle' = 2\sqrt{2}\beta|\alpha^* - \gamma|\cos\xi, \tag{6.30c}$$
$$S_3' = i(\langle\psi|\hat{b}_y^\dagger\hat{b}_x|\rangle' - \langle\psi|\hat{b}_x^\dagger\hat{b}_y|\rangle') = 2\sqrt{2}\beta|\alpha^* - \gamma|\sin\xi. \tag{6.30d}$$

Hence, the Stokes parameters on the screen can be explained in terms of Stokes parameters of the incident field as

$$S_0(\boldsymbol{r},t) = S_0'|f(\boldsymbol{r},t)|^2(1+\cos\Phi), \tag{6.31a}$$
$$S_1(\boldsymbol{r},t) = S_1'|f(\boldsymbol{r},t)|^2(1+\cos\Phi), \tag{6.31b}$$
$$S_2(\boldsymbol{r},t) = S_2'|f(\boldsymbol{r},t)|^2(1+\cos\Phi), \tag{6.31c}$$
$$S_3(\boldsymbol{r},t) = S_3'|f(\boldsymbol{r},t)|^2(1+\cos\Phi). \tag{6.31d}$$

which agrees with the classical result Eq. (4.22) when the intensities are equal at the pinholes. Thus, we have derived the Stokes parameters of the field and used

them to demonstrate the interference of electromagnetic field in view of quantum-dynamics. We also showed that the quantum Stokes parameters at the screen can be expressed in terms of Stokes parameters of the incident field, which suggests that Young's interference experiment is essentially a classical experiment.

Chapter VII

Conclusion

In this thesis, the coherence and the polarization of the fields have been analyzed from classical as well as quantum-mechanical point of view, with a special emphasis on the interference phenomenon in Young's interference experiment. The long-established classical theory easily explains the field coherence and polarizations with field correlation functions and Stokes parameters, which could be employed to investigate the interference between any random fields. Till date, the concept of quantum-mechanical analysis of interference has been limited to scalar fields and we try to extend the concept to encompass a full electromagnetic field.

In analogy to the classical theory, we defined the quantum counterpart of the Stokes parameters in terms of quantum polarization matrix $\mathbf{G}^{(1)}(\boldsymbol{r}, t; \boldsymbol{r}, t)$ which are expressed in terms of expectation values of the field components. Since the interference results from the superposition of radial modes, we exploited it to determine the Stokes parameters at any point $\boldsymbol{r}$ at time t on the screen, that describes the intensity as well the polarization properties of the field. In particular, a two-photon field interference was investigated that showed results parallel to the classical theory; the intensity fringes for a pure state field has distribution similar to one in the classical case, as well as the Stokes parameters of the field on the screen could be explained in terms of the Stokes parameters of the incident field.

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