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MOHAMED AMINE ZEMIRNI

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ACADEMIC DISSERTATION

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Mohamed Amine Zemirni
Results on the asymptotic growth of solutions of complex ODE's
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ABSTRACT

This survey part of the thesis contains new findings concerning complex linear differential equations

$$f^{(n)} + A_{n-1}(z)f^{(n-1)} + \cdots + A_1(z)f' + A_0(z)f = 0,$$

where $n \geq 2$ and $A_0(z), \ldots, A_{n-1}(z)$ are either entire or analytic in the unit disc. Solutions that grow rapidly compared to the coefficients and satisfy certain asymptotic growth properties are shown to have infinite order of growth. This improves, in particular, the classical Frei's theorem and its unit disc counterparts. In the case of second order linear differential equations, new conditions on the coefficients are introduced to ensure that all solutions are of infinite order. The oscillation of solutions in the second order case is also discussed. Moreover, Hille's theory on asymptotic integration is presented in different frameworks.

This survey also contains new results concerning complex nonlinear differential equations

$$f^n + P(z, f) = h(z),$$

where $n \geq 2$, $P(z, f)$ is a differential polynomial in f and its derivatives of degree $\leq n - 1$ with coefficients being small functions of f, and $h(z)$ is a meromorphic solution of a second order linear differential equation with rational coefficients. A result similar to the Tumura-Clunie theorem is given. In the case when $n \geq 3$ and the coefficients of $P(z, f)$ are rationals, the asymptotic growth of solutions of the nonlinear equation above is obtained, and moreover, for a particular case, all possible forms of the meromorphic solutions are given.

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Joensuu, October 1, 2020

Mohamed Amine Zemirni

LIST OF PUBLICATIONS

This thesis consists of the present review of the author's work in the field of complex differential equations and the following selection of the author's publications:

I J. Heittokangas, Z. Latreuch, J. Wang and M. A. Zemirni, "On meromorphic solutions of nonlinear differential equations of Tumura-Clunie type", *to appear in Math. Nachr.*

II J. Heittokangas, H. Yu and M. A. Zemirni, "On the number of linearly independent admissible solutions to linear differential and linear difference equations", *Canad. J. Math.* (2020), 1–35.
`https://doi.org/10.4153/S0008414X20000607`

III J. Heittokangas and M. A. Zemirni, "On Petrenko's deviations and second order differential equations", *to appear in Kodai Math. J.*

IV G. G. Gundersen, J. Heittokangas and M. A. Zemirni, "Asymptotic integration theory for $f'' + P(z)f = 0$", *submitted preprint*
`https://arxiv.org/abs/2008.10262`

Throughout the overview, these papers will be referred to by Roman numerals.

AUTHOR'S CONTRIBUTION

The publications selected in this dissertation are original research papers on complex differential equations, except for Paper IV, which is partially a survey paper.
All authors have made an equal contribution.

TABLE OF CONTENTS

1 Introduction

Throughout this thesis, the asymptotic growth of a function g analytic in D, where D stands for either the complex plane $\mathbb{C}$ or the unit disc $\mathbb{D}$, is meant in general to be one of the following cases:

$$\log M(r,g) \asymp \varphi(r), \quad T(r,g) \asymp \varphi(r) \quad \text{or} \quad \log M(r,g) \asymp T(r,f),$$

where $\varphi(r)$ is an increasing function on either $[0,\infty)$ or $[0,1)$. The asymptotic comparison $x \asymp y$ between two quantities x and y is nothing but $x \lesssim y$ and $x \gtrsim y$, where $x \lesssim y$ means that there exists a constant $M > 0$ for which $x \leq My$, and $x \gtrsim y$ is understood in the same manner. In some cases, we consider the above three cases by replacing $\asymp$ with the asymptotic equivalence notation $\sim$.

In this thesis, we study complex differential equations in terms of: (a) finding the asymptotic growth of solutions and the link to coefficients, and (b) showing the effect of the asymptotic growth of the coefficients on the growth of solutions. The problem (a) is addressed in Paper I regarding nonlinear differential equations, in Paper II regarding linear differential equations, and in Paper IV for particular second order differential equations. Paper III addresses the problem (b) for second order differential equations.

Consider linear differential equations

$$f^{(n)} + A_{n-1}(z)f^{(n-1)} + \cdots + A_1(z)f' + A_0(z)f = 0, \tag{1.1}$$

where the coefficients $A_0(z), \ldots, A_{n-1}(z)$ are either entire functions or analytic functions in $\mathbb{D}$. In the case of their being entire, a classical result of Wittich asserts that all solutions of (1.1) are of finite order if and only if all the coefficients are polynomials [64]. Hence, as a consequence of this result, if one of the coefficients is a transcendental entire function, then (1.1) admits at least one solution of infinite order. This is extended to the classical result of Frei, which addresses more carefully the number of solutions of finite order, or rather the number of solutions of infinite order [12]. The number of solutions of infinite order in Frei's theorem depends on the location of the coefficient that has a maximal growth. Precisely, if $A_p(z)$ is the last transcendental coefficient in the sequence $A_0(z), \ldots, A_{n-1}(z)$, then each solution base of (1.1) contains at least $n-p$ solutions of infinite order. Unit disc counterparts of Wittich's theorem and Frei's theorem are proved by Heittokangas in [24], where the space of polynomials is replaced with the Korenblum space $\mathcal{A}^{-\infty}$. In this thesis, we say that $A_p(z)$ is the last coefficient that has a maximal growth property if its growth dominates the growth of the coefficients $A_{p+1}(z), \ldots, A_{n-1}(z)$, if applicable, in a certain way. In fact, we introduce different ways to express the dominance of $A_p(z)$, and thus we obtain in many cases more accurate results than Frei's theorem. In the complex plane, if the dominance of $A_p(z)$ is expressed in a certain sense by means of the Nevanlinna characteristic, the maximum modulus or along a maximum curve of $A_p(z)$, then each solution base of (1.1) contains at least $n-p$ solutions satisfying $\log T(r,f) \gtrsim \varphi(r)$, where $\varphi(r)$ is either $T(r, A_p)$ or $\log M(r, A_p)$. Specifically, using the maximum modulus leads to $\log T(r,f) \asymp \log M(r, A_p)$. Counterparts of these conclusions in the unit disc are obtained as well. Moreover, in the unit disc case, we give two more ways based on integral means to express the dominance of $A_p(z)$.

In the case of second order differential equations

$$f'' + A(z)f' + B(z)f = 0, \tag{1.2}$$

where $A(z)$ and $B(z)$ are entire functions, we introduce new conditions to ensure that all solutions of (1.2) are of infinite order. These conditions are based on a comparison between the quantities:

$$\xi(A) := \frac{1}{2\pi} \cdot \text{meas} \left(\left\{ \theta \in [0, 2\pi) : \limsup_{r\to\infty} \frac{\log^+ |A(re^{i\theta})|}{\log r} < \infty \right\} \right)$$

and

$$\beta^-(\infty, B) := \liminf_{r\to\infty} \frac{\log M(r,B)}{T(r,B)}.$$

Regarding second order differential equations

$$f'' + A(z)f = 0, \tag{1.3}$$

we show that if $A(z)$ is a transcendental entire function satisfying, in addition to other conditions, the property $T(r,A) \sim \alpha \log M(r,A)$, where $\alpha \in (0,1]$, as $r \to \infty$ outside an exceptional set, then the lower bound for $\max\{\lambda(f_1); \lambda(f_2)\}$ is obtained for any linearly independent solutions f_1 and f_2 of (1.3). This is a generalization of an earlier result by Laine and Wu [41]. When $A(z)$ is a polynomial, Hille's theory on asymptotic integration [32, 33] is presented with updates.

Finally, we consider nonlinear differential equations

$$f^n + P(z,f) = h(z), \quad n \geq 2, \tag{1.4}$$

where $P(z,f)$ is a differential polynomial in f and its derivatives of degree $\leq n-1$ with coefficients being small functions of f, and $h(z)$ is a meromorphic solution of

$$h'' + r_1(z)h' + r_0(z)h = r_2(z), \tag{1.5}$$

where $r_0(z) \not\equiv 0, r_1(z), r_2(z)$ are rational functions. We show that either f has finitely many zeros and poles, or $T(r,f)$ is dominated by the zeros and poles of f in a certain way. This can be seen as an extension of the Tumura-Clunie theorem. Particular attention is addressed to the case when $n \geq 3$ and $P(z,f)$ has rational coefficients, in which the asymptotic growth of meromorphic solutions is given. Consequently, we show the similarity between f and h, and that f satisfies a differential equation with coefficients asymptotically comparable to the coefficients in (1.5).

The remainder of this survey is organized as follows: In Chapter 2, we recall the key notation of Nevanlinna's theory, and discuss some key lemmas such as the lemma on the logarithmic derivatives and its variants. Chapter 3 is devoted to presenting some results concerning complex linear differential equations (LDE's) in the complex plane and in the unit disc, and we discuss, in particular, second order LDE's. We give an overview of the nonlinear differential equations of the form (1.4) in Chapter 4. Finally, the essential contents of Papers I–IV are summarized in Chapter 5.

2 Background on Nevanlinna's theory

Nevanlinna's theory for meromorphic functions is used to obtain the results presented in this dissertation. For the convenience of the reader, we recall the key notation and some fundamentals of this theory. For more details on Nevanlinna's theory and its connection to complex differential equations, we refer to [22,39,69].

Throughout this dissertation, $\mathbb{D} = \{z : |z| < 1\}$ denotes the unit disc of the complex plane $\mathbb{C}$, and the notation $\widehat{\mathbb{C}}$ stands for $\mathbb{C} \cup \{\infty\}$.

2.1 NEVANLINNA CHARACTERISTIC FUNCTION

For a meromorphic function f in $\{z \in: |z| < R^*\}$, where $0 < R^* \leq \infty$, the *Nevanlinna characteristic function* $T(r, f)$ is defined as

$$T(r, f) := m(r, f) + N(r, f),$$

where $m(r, f)$ is the *proximity function* and $N(r, f)$ is the *integrated counting function*, given respectively by

$$m(r, f) := \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f(re^{i\varphi}) \right| d\varphi,$$

$$N(r, f) := \int_0^r \frac{n(t, f) - n(0, f)}{t} \, dt + n(0, f) \log r.$$

Here $\log^+ x := \max\{0, \log x\}$ for $x \geq 0$ and $n(r, f)$ denotes the number of poles of f in $\{z \in \mathbb{C} : |z| \leq r\}, r < R^*$, counting multiplicities. We will also use the notation $\overline{N}(r, f)$ defined by

$$\overline{N}(r, f) := \int_0^r \frac{\overline{n}(t, f) - \overline{n}(0, f)}{t} \, dt + \overline{n}(0, f) \log r,$$

where $\overline{n}(r, f)$ denotes the number of poles of f in $\{z \in \mathbb{C} : |z| \leq r\}$, $r < R^*$, ignoring multiplicities. If f is analytic in $|z| < R^*$, then its characteristic function is simply given as $T(r, f) = m(r, f)$, and in this case, $T(r, f)$ is related to the maximum modulus $M(r, f) = \max\limits_{|z|=r} |f(z)|$ by means of the standard inequalities

$$T(r, f) \leq \log^+ M(r, f) \leq \frac{R + r}{R - r} T(R, f), \tag{2.1}$$

for every $r < R < R^*$. The Nevanlinna characteristic $T(r, f)$ is an increasing convex function of $\log r$ for $0 < r < R^*$ [22, p. 18].

In Nevanlinna's theory, the first main theorem shows that the density of the a-points of f and its average proximity to a are roughly independent of a. The first main theorem is often stated as follows.

Theorem 2.1 ([22, p. 6]). *For any $a \in \mathbb{C}$, we have*

$$T\left(r, \frac{1}{f - a}\right) = T(r, f) + O(1).$$

By appealing to a simple re-scaling, it is enough to consider the cases $R^* = \infty$ and $R^* = 1$ only, that is, we consider functions that are meromorphic in either the complex plane $\mathbb{C}$ or in the unit disc $\mathbb{D}$.

In the case of the complex plane, $T(r, f)$ is an unbounded function of r for any non-constant meromorphic function f. It is known that f is rational if and only if $T(r, f) = O(\log r)$. However, from [69, Theorem 1.5], we know that

$$\lim_{r\to\infty} \frac{T(r, f)}{\log r} = \infty \tag{2.2}$$

when f is a transcendental meromorphic function. The *order of growth* of a meromorphic function f is given by

$$\rho(f) := \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r}.$$

If f is an entire function, then (2.1) allows us to express the order $\rho(f)$ as

$$\rho(f) = \limsup_{r\to\infty} \frac{\log\log M(r, f)}{\log r}.$$

Clunie constructed entire functions having prescribed asymptotic growth [11]. A consequence of his result is that any non-negative real number can be the order of an entire function. Clunie's result reads as follows.

Theorem 2.2 ([11]). *Let $\varphi(r)$ be increasing and convex in $\log r$ with $\varphi(r) \neq O(\log r)$ as $r \to \infty$. Then there is an entire function f such that*

$$T(r, f) \sim \log M(r, f) \sim \varphi(r), \quad r \to \infty.$$

Differing from the complex plane, in the unit disc case, there are unbounded analytic functions f in $\mathbb{D}$ with bounded characteristic $T(r, f)$. A typical example is the function

$$f(z) = \exp\left(\frac{1+z}{1-z}\right), \tag{2.3}$$

which has characteristic $T(r, f) = 1$ for all $r \in [0, 1)$.

For a meromorphic function f in $\mathbb{D}$, the *order of growth* of f is given by

$$\sigma(f) := \limsup_{r\to 1^-} \frac{\log^+ T(r, f)}{\log \frac{1}{1-r}}.$$

Typically, $\sigma(f)$ is called the T-order of growth. If f is analytic in $\mathbb{D}$, then the M-order of growth of f is given by

$$\sigma_M(f) := \limsup_{r\to 1^-} \frac{\log^+ \log^+ M(r, f)}{\log \frac{1}{1-r}}.$$

In this case, we obtain from (2.1) the inequalities

$$\sigma(f) \leq \sigma_M(f) \leq \sigma(f) + 1. \tag{2.4}$$

An analogue of Theorem 2.2 for analytic functions in $\mathbb{D}$ of prescribed asymptotic growth is proved in [47].

2.2 NEVANLINNA DEFICIENCY

For $a \in \widehat{\mathbb{C}}$, we use the notation $m(r,a,f), N(r,a,f)$ and $\overline{N}(r,a,f)$ for $m\left(r, \frac{1}{f-a}\right)$, $N\left(r, \frac{1}{f-a}\right)$ and $\overline{N}\left(r, \frac{1}{f-a}\right)$, respectively, if $a \in \mathbb{C}$, and for $m(r,f), N(r,f)$ and $\overline{N}(r,f)$, respectively, if $a = \infty$. Hence, the first main theorem can be expressed as

$$T(r,f) = m(r,a,f) + N(r,a,f) + O(1). \tag{2.5}$$

The second main theorem is stated as follows.

Theorem 2.3 ([22, p. 31]). *Let f be a non-constant meromorphic function in $\mathbb{C}$, and let $a_1, a_2, \dots, a_q \in \mathbb{C}$ be $q \geq 2$ distinct values. Then*

$$m(r,f) + \sum_{j=1}^{q} m(r,a_j,f) \leq 2T(r,f) - N_1(r) + S(r,f),$$

where $N_1(r) = 2N(r,f) - N(r,f') + N(r,1/f')$, and $S(r,f)$ is a quantity satisfying

$$S(r,f) = O(\log T(r,f) + \log r), \quad r \to \infty, \tag{2.6}$$

outside a possible exceptional set $E \subset [0,\infty)$ of finite linear measure, i.e., $\int_E dt < \infty$.

The *Nevanlinna deficiency*, $\delta(a,f)$, for the a-points of a meromorphic function f is defined by

$$\delta(a,f) := \liminf_{r\to\infty} \frac{m(r,a,f)}{T(r,f)}, \quad a \in \widehat{\mathbb{C}}. \tag{2.7}$$

From (2.5), it is clear that $0 \leq \delta(a,f) \leq 1$, and that $\delta(a,f)$ can also be expressed as

$$\delta(a,f) = 1 - \limsup_{r\to\infty} \frac{N(r,a,f)}{T(r,f)}, \quad a \in \widehat{\mathbb{C}}. \tag{2.8}$$

If $\delta(a,f) > 0$, then a is called a *Nevanlinna deficient value*, or simply a *deficient value*. From (2.8), we deduce that any Picard value of f is also a deficient value of f. A known consequence of the second main theorem is that the set of the deficient values of a meromorphic function is at most countable [22, p. 43]. If f is a meromorphic function with lower order $\mu(f) = 0$, where

$$\mu(f) := \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r},$$

then f cannot have more than one deficient value [14, p. 201]. If f is an entire function with $\mu(f) \leq 1/2$, then f cannot have finite deficient values [14, p. 207].

We recall

$$\Theta(a,f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,a,f)}{T(r,f)}, \quad a \in \widehat{\mathbb{C}}. \tag{2.9}$$

It is clear that $0 \leq \delta(a,f) \leq \Theta(a,f) \leq 1$. Moreover, the set of the values a for which $\Theta(a,f) > 0$ is at most countable, and for any meromorphic function f, the *defect relation* is known as

$$\sum_{a\in\widehat{\mathbb{C}}} \delta(a,f) \leq \sum_{a\in\widehat{\mathbb{C}}} \Theta(a,f) \leq 2. \tag{2.10}$$

This is also a consequence of the second main theorem [22, p. 43].

Differing from the plane case, in the unit disc we need to assume that $T(r,f)$ is unbounded, and the quantity $S(r,f)$ in Theorem 2.3 satisfies

$$S(r,f) = O\left(\log T(r,f) + \log \frac{1}{1-r}\right), \quad r \to 1^-, \tag{2.11}$$

outside a possible exceptional set $E \subset [0,1)$ with $\int_E \frac{dt}{1-t} < \infty$. The Nevanlinna deficiency for the a-points of a meromorphic function f in $\mathbb{D}$ is defined analogously as in the case $\mathbb{C}$ simply by replacing "$r \to \infty$" with "$r \to 1^-$" [58].

2.3 LOGARITHMIC DERIVATIVES

Estimating the growth of logarithmic derivatives of meromorphic functions is very essential in establishing many results in the value distribution theory, such as, the second main theorem. In fact, the error term $S(r,f)$ in Theorem 2.3 arises from estimating the logarithmic derivatives by means of the proximity function. This last estimation is called the standard lemma on the logarithmic derivatives and reads as follows.

Theorem 2.4 ([39, p. 36]). *Let f be a meromorphic function and $k \geq 1$ be an integer. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r,f),$$

where $S(r,f)$ satisfies (2.6).

For meromorphic functions in $\mathbb{D}$, we have the same lemma on the logarithmic derivatives but $S(r,f)$ satisfies (2.11) [24, p. 8].

For applications in the theory of complex differential equations, estimating the logarithmic derivatives is indispensable due to the appearance of a function and its derivatives in the same equation. In addition, we need to estimate the logarithmic derivatives by different means other than the proximity function. Below are some estimates stated separately in the complex plane case and the unit disc case.

Complex plane

The following result due to Gundersen is very useful in the theory of complex differential equations.

Theorem 2.5 ([16]). *Let f be a transcendental meromorphic function, and let $\alpha > 1$ be a given real constant. For any integers k, j with $k > j \geq 0$, the following statements hold.*

(i) There exists a set $E \subset [0, 2\pi)$ that has linear measure zero, and there exists a constant $B > 0$, such that if $\varphi \in [0, 2\pi) \setminus E$, then there is a constant $R_0 = R_0(\varphi) > 0$ such that for all z satisfying $\arg(z) = \varphi$ *and $|z| > R_0$, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leqslant B\left(\frac{T(\alpha r, f)}{r}(\log r)^{\alpha} \log T(\alpha r, f)\right)^{k-j}. \tag{2.12}$$

(ii) There is a set $F \subset (1, \infty)$ that has finite logarithmic measure, i.e., $\int_F dt/t < \infty$, and there is a constant $B > 0$, such for all z satisfying $|z| \notin (F \cup [0,1])$, we have (2.12).

In the case when f is of finite order, the following consequence of Theorem 2.5 seems to be very useful as well.

Corollary 2.1 ([16]). *Let f be a transcendental meromorphic function of finite order ρ, and let $\varepsilon > 0$ be a given real constant. For any integers k, j with $k > j \geq 0$, the following statements hold.*

(i) There exists a set $E \subset [0, 2\pi)$ that has linear measure zero, and there exists a constant $B > 0$, such that if $\varphi \in [0, 2\pi) \setminus E$, then there is a constant $R_0 = R_0(\varphi) > 0$ such that for all z satisfying $\arg(z) = \varphi$ *and $|z| > R_0$, we have*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leqslant |z|^{(k-j)(\rho-1+\varepsilon)}. \tag{2.13}$$

(ii) There is a set $F \subset (1, \infty)$ that has finite logarithmic measure, and there is a constant $B > 0$, such for all z satisfying $|z| \notin (F \cup [0, 1])$, we have (2.13).

Unit disc

The unit disc counterpart of Theorem 2.5 can be seen as Theorem 3.1 in [7]. The unit disc counterpart of Corollary 2.1 by means of T-order is [7, Corollary 3.2], and by means of M-order is [8, Corollary 1.3]. The exceptional set appearing in [8, Corollary 1.3] is of arbitrary small upper final density. Recall that the upper final density of a set $E \subset [0, 1)$ is given by

$$\overline{d}(E) := \limsup_{r \to 1^-} \frac{1}{1-r} \int_{E \cap [r, 1)} dr.$$

It is clear that $0 \leq \overline{d}(E) \leq 1$ for any set $E \subset [0, 1)$.

The following two theorems are recent results concerning the logarithmic derivatives [6]. In the original statements in [6], f is meromorphic in a disc $D(0, R)$. Here, for simplicity, f is meromorphic in $\mathbb{D}$.

Theorem 2.6 ([6, Corollary 7]). *Let f be meromorphic in $\mathbb{D}$, and let k, j be integers with $k > j \geq 0$ such that $f^{(j)} \not\equiv 0$. Let $s : [0, 1) \to [0, 1)$ be an increasing continuous function such that $s(r) \in (r, 1)$ and $s(r) - r$ is decreasing. If $\delta \in (0, 1)$, then there exists a measurable set $E \subset [0, 1)$ with $\overline{d}(E) \leq \delta$ such that*

$$\int_0^{2\pi} \left| \frac{f^{(k)}(re^{i\theta})}{f^{(j)}(re^{i\theta})} \right|^{\frac{1}{k-j}} d\theta \lesssim \frac{T(s(r), f) - \log(s(r) - r)}{s(r) - r}, \quad r \notin E. \tag{2.14}$$

Moreover, if $k = 1$ and $j = 0$, then the logarithmic term in (2.14) *can be omitted.*

The following theorem is a new estimate on logarithmic derivatives.

Theorem 2.7 ([6, Corollary 6]). *Let f be meromorphic in $\mathbb{D}$. Suppose that k, j are integers with $k > j \geq 0$, and $f^{(j)} \not\equiv 0$. Then, for $0 \leq r' < r < R < 1$,*

$$\begin{aligned} &\int_{r' < |z| < r} \left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right|^{\frac{1}{k-j}} dm(z) \\ &\qquad \lesssim R \log \frac{e\,(R - r')}{R - r} \left(1 + \log^+ \frac{1}{R - r} + T(R, f) \right). \end{aligned} \tag{2.15}$$

Here, $dm(z)$ is the Lebesgue measure in $r' < |z| < r$.

When using these estimates in proving Theorems 5.9 and 5.10 below, we need to reduce $s(r)$ in Theorem 2.6 and R in Theorem 2.7 to r. This reduction can be done by means of Borel's lemma [22, Lemma 2.4]. Therefore, by taking $s(r) = R = r + (1-r)/(eT(r,f))$ and by using Borel's lemma, we get the following two estimations parallel to (2.14) and (2.15):

$$\log^+ \int_0^{2\pi} \left| \frac{f^{(k)}(re^{i\theta})}{f^{(j)}(re^{i\theta})} \right|^{\frac{1}{k-j}} d\theta \lesssim \log T(r,f) + \log \frac{1}{1-r}, \quad r \notin E,$$

where $E \subset [0,1)$ is a set with $\overline{d}(E) < 1$, and

$$\log^+ \int_{D(0,r)} \left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right|^{\frac{1}{k-j}} dm(z) \lesssim \log T(r,f) + \log \frac{1}{1-r}, \quad r \notin E,$$

where $\int_E \frac{dt}{1-t} < \infty$.

3 Complex linear differential equations

This chapter is devoted to presenting some selected results on linear differential equations of the form

$$f^{(n)} + A_{n-1}(z)f^{(n-1)} + \cdots + A_1(z)f' + A_0(z)f = 0, \tag{3.1}$$

where $n \geq 2$ is an integer, and $A_0(z) \not\equiv 0$. If all the coefficients $A_0(z), \ldots, A_{n-1}(z)$ are analytic in a simply connected domain $D \subset \mathbb{C}$, then all the solutions of (3.1) are analytic in D as well [40]. In particular, if $A_0(z), \ldots, A_{n-1}(z)$ are entire, then all the solutions of (3.1) are entire. Moreover, the zeros of any solution are of multiplicity at most $n-1$. Conversely, if f is a given non-zero entire function whose zeros all have multiplicity at most $n-1$, then f is a solution of some differential equation of the form (3.1) with entire coefficients [21, p. 300].

If the coefficients $A_0(z), \ldots, A_{n-1}(z)$ are meromorphic functions in $\mathbb{C}$, then solutions of (3.1) may not always be meromorphic in $\mathbb{C}$. For example, the functions $\exp((z-a)^{-1})$ and $\exp(-(z-a)^{-1})$, where $a \in \mathbb{C}$, are non-meromorphic in $\mathbb{C}$, but they satisfy the differential equation

$$f'' + \frac{2}{z-a}f' - \frac{1}{(z-a)^4}f = 0.$$

It may also happen that some solutions are meromorphic while others are not. For example, the differential equation

$$f'' - \frac{1}{z}f' + \frac{1}{z^2}f = 0$$

has two linearly independent solutions $f_1(z) = z$ and $f_2(z) = z\log(z)$. Here, f_1 is entire function, while f_2 is non-meromorphic in $\mathbb{C}$.

3.1 LDE'S IN THE COMPLEX PLANE

In this section and in the next, we present some results related mainly to Paper II. We begin with the following known theorem due to Wittich.

Theorem 3.1 ([39, Theorem 4.1]). *Let the coefficients $A_0(z), \ldots, A_{n-1}(z)$ in* (3.1) *be entire functions. Then $A_0(z), \ldots, A_{n-1}(z)$ are polynomials if and only if all solutions of* (3.1) *are entire functions of finite order.*

For any transcendental solution f of (3.1) with polynomial coefficients, we have

$$\frac{1}{n-1} \leq \rho(f) \leq 1 + \max_{0\leq j\leq n-1} \frac{\deg A_j}{n-j}.$$

The left inequality is in [20, Corollary 2], while the right inequality is in [39, Proposition 7.1]. For the corresponding rational coefficients case, a transcendental meromorphic solution f of (3.1) with rational coefficients satisfies

$$\frac{1}{n} \leq \rho(f) \leq 1 + \max_{0\leq j\leq n-1} \frac{\deg_\infty A_j}{n-j},$$

where $\deg_\infty A_j$ is the degree of A_j at infinity. Here, the left inequality can be found in [35, Satz 22.1] and the right inequality is proved in [39, Proposition 7.2].

Concerning the possible orders, Wittich proved the following result.

Theorem 3.2 ([65, pp. 65–67]). *For a given equation of the form* (3.1) *with polynomial coefficients, there exist $p \leq n$ positive rational numbers $\alpha_1, \dots, \alpha_p$, such that if f is any transcendental solution of* (3.1)*, then $\rho(f) \in \{\alpha_1, \dots, \alpha_p\}$.*

The conclusion of Theorem 3.2 holds for non-homogeneous linear differential equations with rational coefficients, see for example [35, Satz 22.1].

The explicit forms for the possible orders in Theorem 3.2 are given in [20]. To present these forms here, we recall the following construction. For $j = 0, \dots, n-1$, set

$$d_j = \begin{cases} \deg(A_j), & \text{if } A_j \not\equiv 0, \\ -\infty, & \text{if } A_j \equiv 0. \end{cases}$$

Next, a sequence of integers $\{s_k\}_k$ is defined as follows. Define s_1 by

$$s_1 = \min\left\{ j : \frac{d_j}{n-j} = \max_{0 \leq l \leq n-1} \frac{d_l}{n-l} \right\};$$

and if s_k is constructed, then s_{k+1} is defined as

$$s_{k+1} = \min\left\{ j : \frac{d_j - d_{s_k}}{s_k - j} = \max_{0 \leq l < s_k} \frac{d_l - d_{s_k}}{s_k - l} > -1 \right\}.$$

Note that s_1 always exists, and s_k may not exist for $k > 1$. Actually, there exists some positive integer $p \leq n$, such that the integers $s_1, \dots, s_p$ exist and s_{p+1} does not exist. From this construction, it is obvious that $s_1 > \cdots > s_p \geq 0$. Correspondingly, define

$$\alpha_k = 1 + \frac{d_{s_k} - d_{s_{k-1}}}{s_{k-1} - s_k}, \quad k = 1, \dots, p, \tag{3.2}$$

where $s_0 := n$ and $d_{s_0} := 0$. It is proved in [20, Theorem 1] that these α_k, $k = 1, \dots, p$, are the possible orders mentioned in Theorem 3.2.

Regarding the asymptotic growth of transcendental solutions of (3.1) with polynomial coefficients, Valiron proves that each transcendental entire solution f of (3.1) with rational coefficients has the asymptotic property

$$\log M(r, f) \sim Cr^\rho, \quad r \to \infty, \tag{3.3}$$

where $C > 0$ is some constant and ρ is a positive rational number [63, pp. 106–108]. In fact, ρ takes one value α_k from the list (3.2). All possible values for the constant C in the case of second order differential equations are given in Paper I, see Lemma 5.1 below.

Suppose that all the coefficients $A_0(z), \dots, A_{n-1}(z)$ in (3.1) are entire functions and at least one of them is transcendental. Then from Theorem 3.1, the equation (3.1) admits at least one solution of infinite order. In this context, the question about the number of solutions of the equation (3.1) with infinite order is of interest. The classical theorem of Frei is one of the seminal results regarding this question, and it reads as follows.

Theorem 3.3 ([39, p. 60])**.** *Suppose that the coefficients in* (3.1) *are entire, and that at least one of them is transcendental. Suppose that $A_p(z)$ is the last transcendental coefficient while the coefficients $A_{p+1}(z), \ldots, A_{n-1}(z)$, if applicable, are polynomials. Then every solution base of* (3.1) *has at least $n-p$ solutions of infinite order.*

In the value distribution of the solutions of (3.1), Theorem 3.4 below says that the value 0 has a special interest. The original statement of Theorem 3.4 is for rational coefficients [65, p. 54], but the conclusion remains true with small meromorphic coefficients.

Theorem 3.4 ([39, p. 62])**.** *Suppose that a meromorphic solution f of* (3.1) *is an admissible solution in the sense that*

$$T(r, A_j) = S(r, f), \quad j = 0, \ldots, n-1. \tag{3.4}$$

Then 0 *is the only possible finite Nevanlinna deficient value for f.*

3.2 LDE'S IN THE UNIT DISC

Regarding the differential equation (3.1) in the unit disc $\mathbb{D}$, we recall first the definition of the *Korenblum space* $\mathcal{A}^{-\infty}$, introduced in [37], which is defined as

$$\mathcal{A}^{-\infty} := \bigcup_{q \geq 0} \mathcal{A}^{-q},$$

where $\mathcal{A}^{-q}$ is a function space defined as

$$\mathcal{A}^{-q} := \left\{ g : g \text{ is analytic in } \mathbb{D} \text{ and } \sup_{z \in \mathbb{D}} (1 - |z|^2)^q |g(z)| < \infty \right\}.$$

Analytic functions belonging to $\mathcal{A}^{-\infty}$ can assume the role of the polynomials.

Theorem 3.5 below is an analogue of Theorem 3.1 for the unit disc $\mathbb{D}$.

Theorem 3.5 ([24, Theorem 6.1])**.** *Let the coefficients $A_0(z), \ldots, A_{n-1}(z)$ in* (3.1) *be analytic in $\mathbb{D}$. Then $A_0(z), \ldots, A_{n-1}(z)$ belong to $\mathcal{A}^{-\infty}$ if and only if all non-trivial solutions of* (3.1) *are analytic and of finite order of growth in $\mathbb{D}$.*

If at least one of the coefficients $A_0(z), \ldots, A_{n-1}(z)$ does not belong to $\mathcal{A}^{-\infty}$, then from Theorem 3.5, the equation (3.1) must have at least one solution with infinite order. A counterpart of Frei's theorem in $\mathbb{D}$ is given as follows.

Theorem 3.6 ([24, Theorem 6.3])**.** *Suppose that the coefficients $A_0(z), \ldots, A_{n-1}(z)$ in* (3.1) *are analytic in $\mathbb{D}$, and that at least one of them is not in $\mathcal{A}^{-\infty}$. Suppose that $A_p(z)$ is the last coefficient not being in $\mathcal{A}^{-\infty}$, while the coefficients $A_{p+1}(z), \ldots, A_{n-1}(z)$, if applicable, are in $\mathcal{A}^{-\infty}$. Then every solution base of* (3.1) *has at least $n-p$ solutions of infinite order.*

Note that the order of growth of solutions in Theorem 3.6 is not specified for either T-order or M-order. The conclusion in Theorem 3.6 actually works for both growth orders due to the inequalities in (2.4).

Theorem 3.6 is considered as the first formulation of Frei's theorem in $\mathbb{D}$, where the growth of the coefficients is estimated in terms of the maximum modulus. By estimating the growth of the coefficients in terms of the Nevanlinna characteristic, we obtain the second formulation of Frei's theorem in $\mathbb{D}$. Before stating the second formulation, we recall that a meromorphic function f in $\mathbb{D}$ is called *admissible* if

$$\limsup_{r\to 1^-} \frac{T(r,f)}{-\log(1-r)} = \infty, \tag{3.5}$$

otherwise f is called *non-admissible*. In fact for an admissible function f satisfying (3.5), we can prove, using a technical lemma from [70, p. 13], that there exists a set $F \subset [0,1)$ with $\overline{ld}(F) = 1$ such that

$$\lim_{\substack{r\to 1^- \\ r\in F}} \frac{T(r,f)}{-\log(1-r)} = \infty.$$

This is a unit disc analogue of (2.2). Here, $\overline{ld}(F)$ is the upper logarithmic density of the set $F \subset [0,1)$, and it is defined as

$$\overline{ld}(F) = \limsup_{r\to 1^-} \frac{1}{-\log(1-r)} \int_{F\cap[0,r)} \frac{dt}{1-t}.$$

Theorem 3.7 ([28])**.** *Suppose that the coefficients $A_0(z), \ldots, A_{n-1}(z)$ in* (3.1) *are analytic in $\mathbb{D}$, and that at least one of them is admissible. Suppose that $A_p(z)$ is the last admissible coefficient while the coefficients $A_{p+1}(z), \ldots, A_{n-1}(z)$, if applicable, are non-admissible. Then every solution base of* (3.1) *has at least $n-p$ solutions of infinite order.*

The direct proof of Theorem 3.7 is very similar to the proof of Theorem 3.6. However, Theorem 3.7 follows as a special case of the results in Paper II.

If $f \in \mathcal{A}^{-\infty}$, then f is non-admissible. However, there are non-admissible functions not belonging to $\mathcal{A}^{-\infty}$, for example, the function f in (2.3). This shows the difference between the two formulations of Frei's theorem in $\mathbb{D}$.

In the unit disc, we use the term "admissible" with two different meanings . The second meaning arises from the following unit disc analogue of Theorem 3.4.

Theorem 3.8. *Suppose that a meromorphic solution f in $\mathbb{D}$ of* (3.1) *is an admissible solution in the sense that*

$$T(r, A_j) = S(r,f), \quad j = 0, \ldots, n-1. \tag{3.6}$$

Then 0 *is the only possible finite deficient value for f.*

To differentiate between the two meanings of "admissible", the term "admissible" used in the sense (3.6) will always be followed by the term "solution".

3.3 THE ORDER REDUCTION METHOD

The standard order reduction method ([39, pp. 60–61], [20, p. 1233]) reduces the nth order linear differential equation (3.1) to a linear differential equation of order $q \leq n$. This method is the main step in proving several theorems regarding the growth of solutions of (3.1), such as Frei's theorem in the complex plane and in the unit disc.

Since this subsection is of independent interest, we will assume in this particular part of the dissertation that the coefficients of (3.1) are meromorphic in a simply connected domain D. Rename the coefficients $A_0(z), \ldots, A_{n-1}(z)$ in (3.1) by $A_{0,0}(z), \ldots, A_{0,n-1}(z)$, and let $f_{0,1}, \ldots, f_{0,n}$ be linearly independent solutions of (3.1).

Then, the order reduction method asserts that the functions $f_{q,s}$ defined by

$$f_{q,s} = \left(\frac{f_{q-1,s+1}}{f_{q-1,1}} \right)', \quad 1 \le q \le n-1, \ 1 \le s \le n-q, \tag{3.7}$$

are linearly independent solutions of the equation

$$f^{(n-q)} + A_{q,n-q-1}(z) f^{(n-q-1)} + \cdots + A_{q,0}(z) f = 0,$$

where

$$A_{q,j}(z) = A_{q-1,j+1}(z) + \sum_{k=j+2}^{n-q+1} \binom{k}{j+1} A_{q-1,k}(z) \frac{f_{q-1,1}^{(k-j-1)}}{f_{q-1,1}}, \quad j = 0, \ldots, n-q-1.$$

In the complex plane case and in the unit disc case, estimating the Nevanlinna characteristic of the functions $f_{q,s}$ defined in (3.7) is needed to prove the main results in Paper II.

Lemma 3.1. *Suppose that $f_{0,1}, \ldots, f_{0,n}$ are linearly independent meromorphic functions in $\mathbb{C}$. Define functions $f_{q,s}$ as in* (3.7). *Then*

$$T(r, f_{q,s}) \lesssim \sum_{l=1}^{q+s} T(r, f_{0,l}) + \log r, \quad r \notin E,$$

where $E \subset [0, \infty)$ is a set of finite linear measure.

Lemma 3.1 appears in [23, p. 234] with a slight modification in (3.7). A unit disc counterpart of Lemma 3.1 is stated similarly, where $\log r$ is replaced with $\log \frac{1}{1-r}$ and the set E satisfies $E \subset [0, 1)$ with $\int_E \frac{dr}{1-r} < \infty$.

The following lemma is based on the order reduction method. It is proved in Paper II ([28, Lemma 4.3]), even though it is mentioned in the proof of Theorem 5.6 in [23, p. 244], but without giving the precise form of the differential polynomials C_k in (3.9). It turns out that the exact form (3.9) is needed in proving our results.

Lemma 3.2. *Let the coefficients $A_0(z), \ldots, A_{n-1}(z)$ in* (3.1) *be meromorphic functions in a simply connected domain D, and let $f_{0,1}, \ldots, f_{0,n}$ be linearly independent solutions of* (3.1). *Define functions $f_{q,s}$ as in* (3.7). *Then, for $p \in \{0, 1, \ldots, n-1\}$, we have*

$$-A_p = C_n + A_{n-1} C_{n-1} + \cdots + A_{p+1} C_{p+1}, \tag{3.8}$$

where $C_{p+1}, \ldots, C_n$ have the following form

$$C_k = \sum_{l_0 + l_1 + \cdots + l_p = k-p} K_{l_0, l_1, \ldots, l_p} \frac{f_{0,1}^{(l_0)}}{f_{0,1}} \frac{f_{1,1}^{(l_1)}}{f_{1,1}} \cdots \frac{f_{p,1}^{(l_p)}}{f_{p,1}}, \quad p+1 \le k \le n. \tag{3.9}$$

Here $0 \le l_0, l_1, \ldots, l_p \le k-p$ and $K_{l_0, l_1, \ldots, l_p}$ are absolute positive constants.

3.4 LDE'S OF SECOND ORDER

In this section, we shed some light on results related to the problems treated in Papers III and IV concerning the growth and oscillation of the second order differential equations

$$f'' + A(z)f = 0 \tag{3.10}$$

and

$$f'' + a(z)f' + b(z)f = 0, \tag{3.11}$$

where $A(z)$, $a(z)$ and $b(z)$ are entire functions.

Equation (3.10) with a polynomial coefficient

If $A(z)$ is a constant, then (3.10) can be solved explicitly, and hence there is no need to discuss this case. Therefore, we consider $A(z)$ to be a polynomial

$$A(z) = p_n z^n + p_{n-1} z^{n-1} + \cdots + p_0, \quad p_n \neq 0,\ n \geq 1. \tag{3.12}$$

Then every non-trivial solution f of (3.10) satisfies $\rho(f) = (n+2)/2$ [2, Theorem 1]. A more precise result is the following asymptotic equality [15, Theorem 6]

$$\log M(r,f) \sim \frac{2\sqrt{|p_n|}}{n+2}\, r^{(n+2)/2}, \quad r \to \infty. \tag{3.13}$$

In addition, if E is a product of two linearly independent solutions of (3.10), then $\lambda(E) = \rho(E) = (n+2)/2$, see [39, Chapter 5]. Here $\lambda(g)$ denotes *the exponent of convergence* of the zero sequence of g, and it is defined by

$$\lambda(g) := \limsup_{r\to\infty} \frac{\log N(r,1/g)}{\log r}.$$

Regarding the location of the zeros of solutions f of (3.10), it is shown in [32, Chapter 7.4] that all but finitely many zeros of f lie in $n+2$ sectors

$$W_j(\varepsilon) = \{z : |\arg(z) - \theta_j| < \varepsilon\}, \quad \theta_j = \frac{2\pi j - \arg(p_n)}{n+2}, \tag{3.14}$$

where $-\pi < \arg(p_n) \leq \pi$, $\varepsilon > 0$ is arbitrarily small and $j = 0, \ldots, n+1$.

A ray $\arg(z) = \theta_j$, where θ_j is given in (3.14) for some $j \in \{0, \ldots, n+1\}$, is called a *critical ray* of the equation (3.10). If a solution f of (3.10) has only finitely many zeros in a sector $W_j(\varepsilon)$ around the critical ray $\arg(z) = \theta_j$, then this critical ray is called a *shortage ray* [15]. The number of shortage rays of a solution f is called the shortage of f and is denoted by $s(f)$. It is clear that $0 \leq s(f) \leq n+2$ for any solution f of (3.10). It is proved in [15, Theorem 5] that if f is a solution of (3.10), then $s(f)$ is an even number and

$$T(r,f) \sim \frac{4(n+2) - 2s(f)}{\pi(n+2)^2} \sqrt{|p_n|}\, r^{(n+2)/2}, \quad r \to \infty. \tag{3.15}$$

Equation (3.10) **with a transcendental coefficient**

Any non-trivial solution f of (3.10) with a transcendental coefficient is of infinite order. Meanwhile, the exponent of convergence of zeros $\lambda(E)$ of the product E is not always infinite. Here, we mention a few results concerning $\lambda(E)$.

We summarize results by Bank-Laine, Rossi and Shen.

Theorem 3.9 ([2,3,56,57]). *Let $A(z)$ be a transcendental entire function, and let E be a product of two linearly independent solutions of* (3.10). *Then the following assertions hold.*

1. *If $\rho(A) \notin \mathbb{N}$, then*
$$\lambda(E) \geq \rho(A). \tag{3.16}$$

2. *If $1/2 < \rho(A) < 1$, then*
$$\lambda(E) \geq \frac{\rho(A)}{2\rho(A) - 1}. \tag{3.17}$$

3. *If $\rho(A) \leq 1/2$, then $\lambda(E) = \infty$.*

When the order of $A(z)$ is replaced with its lower order, results improving the inequalities (3.16) and (3.17) can be found, e.g., in [34,59] and [5, Theorem 1.3].

It is conjectured by Bank and Laine [2,3] that $\lambda(E) = \infty$ whenever $\rho(A) \notin \mathbb{N}$. This conjecture is false, in general, as shown by Bergweiler and Eremenko in the following result.

Theorem 3.10 ([4,5]). *(i) There is a dense set of $\rho \geq 1$, such that there exists an entire function $A(z)$ with $\rho(A) = \lambda(E) = \rho$.*

(ii) For any $\rho \in (1/2, 1)$ there exists an entire function $A(z)$ of order ρ such that
$$\lambda(E) = \frac{\rho(A)}{2\rho(A) - 1}.$$

In the results mentioned above, the lower bound of $\lambda(E)$ depends on either $\rho(A)$ or $\mu(A)$. However, the following result, shows that the asymptotic growth of $A(z)$ has a strong affect on $\lambda(E)$ without taking into account $\rho(A)$ or $\mu(A)$.

Theorem 3.11 ([41]). *Let A be a transcendental entire function of finite order satisfying*
$$T(r, A) \sim \log M(r, A), \quad r \to \infty, \tag{3.18}$$
outside an exceptional set G of finite logarithmic measure. If E is a product of two linearly independent solutions of (3.10), *then $\lambda(E) = \infty$.*

An example of functions satisfying (3.18) outside an exceptional set G of finite logarithmic measure is entire functions $A(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ with Fejér gaps, i.e., $\sum \lambda_n^{-1} < \infty$. Moreover, the existence of entire functions satisfying (3.18) without an exceptional set is guaranteed by Theorem 2.2.

Equation (3.11)

Let $u(z)$ be a primitive function of $-\frac{1}{2}a(z)$, i.e., $u'(z) = -\frac{1}{2}a(z)$, and let f be a non-trivial solution of (3.11). Then $w(z) = f(z)e^{-u(z)}$ is a solution of equation (3.10) with $A(z) = b(z) - \frac{1}{4}a(z)^2 - \frac{1}{2}a'(z)$, see [39, p. 74]. This transformation shows that the oscillation theory of (3.11) is equivalent to that of (3.10). Hence, we restrict ourselves to investigating only the growth of solutions for an equation of the form (3.11).

If either $a(z)$ or $b(z)$ is transcendental, then from Theorem 3.1, equation (3.11) has at least one solution of infinite order. The equation (3.11) can also have finite order solutions; for example, $f(z) = e^{-z}$ solves (3.11) with $a(z) = e^z$ and $b(z) = e^z - 1$. This in fact leads to the question [17]: *What conditions on $a(z)$ and $b(z)$ will guarantee that every solution $f \not\equiv 0$ of* (3.11) *has infinite order?* Examples of such conditions are:

(i) $\rho(a) < \rho(b)$,

(ii) $\rho(b) < \rho(a) \leq 1/2$,

(iii) $a(z)$ is a polynomial and $b(z)$ is transcendental,

(iv) $a(z)$ is transcendental with $\rho(a) = 0$ and $b(z)$ is a polynomial;

see Theorems 2 and 6 in [17] and the main result in [29].

Additionally, many conditions other than those on the growth of $a(z)$ and $b(z)$ have been found, see [49] and the references therein.

The following result by Laine and Wu is a typical example of conditions that do not restrict the growth of the coefficients; moreover, it falls within the same context of this dissertation.

Theorem 3.12 ([42])**.** *Suppose that $a(z)$ and $b(z) \not\equiv 0$ are entire functions such that $\rho(b) < \rho(a) < \infty$ and*

$$T(r,a) \sim \log M(r,a), \quad r \to \infty,$$

outside a set G of finite logarithmic measure. Then every non-trivial solution of (3.11) *is of infinite order.*

Kwon and Kim [38] improved Theorem 3.12 by letting the set G satisfy $\overline{\text{logdens}}(G) < (\rho(a) - \rho(b))/\rho(a)$. Here, the upper logarithmic density $\overline{\text{logdens}}(G)$ is defined by

$$\overline{\text{logdens}}(G) := \limsup_{r\to\infty} \frac{1}{\log r} \int_{G\cap[1,r]} \frac{dt}{t}.$$

The following result is a generalized version of Theorem 3.12.

Theorem 3.13 ([51, Theorem 1.5])**.** *Suppose that $a(z)$ an $b(z)$ are entire functions such that $\mu(b) < \mu(a) < \infty$ and*

$$T(r,a) \sim \alpha \log M(r,a), \quad r \to \infty, \tag{3.19}$$

outside a set G satisfying $\overline{\text{logdens}}(G) = 0$, where $\alpha \in (0,1]$. Then every non-trivial solution f of (3.11) *satisfies*

$$\rho(f) \geq \frac{\mu(a) - \mu(b)}{21(\mu(a) + \mu(b))\sqrt{2\pi(1-\alpha)}} - 1.$$

In particular, if $\alpha = 1$, then $\rho(f) = \infty$.

As mentioned in [51], the condition (3.19) is quite natural. An example of functions $a(z)$ satisfying (3.19) with an exceptional set is the exponential polynomial of the form
$$a(z) = P_1(z)e^{Q_1(z)} + \cdots + P_n(z)e^{Q_n(z)},$$
where $P_j(z)$ and $Q_j(z)$ are polynomials. [51, Example 2.3] shows that $a(z)$ satisfies (3.19) for
$$\alpha = \frac{\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} h_a^+(\theta)d\theta}{\displaystyle\max_{0\leq\theta\leq 2\pi} h_a(\theta)},$$
outside a set G with $\overline{\text{logdens}}(G) = 0$. Here $h_a(\theta)$ is the Phragmén-Lindelöf indicator function of $a(z)$, and $h_a^+(\theta) = \max\{0; h_a(\theta)\}$.

To give an example for functions $a(z)$ satisfying (3.19) without an exceptional set, let $a(z)$ be a solution of (3.10), where $A(z)$ is a polynomial given as in (3.12). Then (3.13) and (3.15) imply that $a(z)$ satisfies (3.19) with
$$\alpha = \frac{2(n+2) - s(a)}{\pi(n+2)},$$
without an exceptional set. Based on this, we can see, for example, that the Airy integral $Ai(z)$ satisfies (3.19) with $\alpha = \frac{4}{3\pi}$ without an exceptional set. Recall that the Airy integral $Ai(z)$ is a contour integral solution of the differential equation $f'' - zf = 0$. It is clear from (3.14) that $Ai(z)$ has three critical rays. Since all the zeros of $Ai(z)$ lie on the negative real axis, which is one of the critical rays for $Ai(z)$, it follows that Ai has two shortage rays, i.e., $s(Ai) = 2$. For generalized Airy functions we refer to [19].

The idea of studying equations of the form (3.11) with coefficients satisfying equations of the form (3.10) with a polynomial coefficient is used, namely in [50, 52, 53, 66]. A similar idea is used in Paper I to study nonlinear differential equations.

4 Complex nonlinear differential equations

In this chapter, we offer a glimpse of recent results concerning the transcendental meromorphic solutions f of nonlinear differential equations of the form

$$f^n + P(z, f) = h(z), \tag{4.1}$$

where $n \geq 2$ is an integer, h is a meromorphic function, and $P(z, f)$ is a differential polynomial in f and its derivatives with coefficients $a(z)$ being small functions of f.

The equation (4.1) in this form appears in the following result by Hayman.

Theorem 4.1 ([22, p. 69]). *Let f and h be non-constant meromorphic functions in $\mathbb{C}$ satisfying the equality* (4.1), *where $P(z, f)$ is a differential polynomial of degree at most $n-1$ in f and its derivatives. If the coefficients of $P(z, f)$ are small functions of f, and if*

$$N(r, f) + N\left(r, \frac{1}{h}\right) = S(r, f),$$

then $h(z) = (f(z) + \alpha(z))^n$, where α is a small function of f.

Theorem 4.1 is an extension of the Tumura-Clunie theorem [10,61]. The following lemma due to Clunie is used to prove Theorem 4.1 as well as other extensions for the Tumura-Clunie theorem.

Lemma 4.1 ([39, p. 39]). *Let f be a transcendental meromorphic solution of*

$$f^n Q^*(z, f) = Q(z, f),$$

where $Q^*(z, f)$ and $Q(z, f)$ are polynomials in f and its derivatives with meromorphic coefficients, say $\{a_\lambda : \lambda \in I\}$, such that $m(r, a_\lambda) = S(r, f)$ for all $\lambda \in I$. If the total degree of $Q(z, f)$ as a polynomial in f and its derivatives is $\leq n$, then

$$m(r, Q^*(z, f)) = S(r, f).$$

Looking at the proof of Lemma 4.1 [39, p. 40], we can get

$$m(r, Q^*(z, f)) = O(\log r),$$

provided that f is of finite order and the coefficients of $Q^*(z, f)$ and $Q(z, f)$ are rational functions.

In the early 2000's, the research regarding the form and the number of transcendental meromorphic solutions of (4.1), where h is a given meromorphic function, became more active. For example, Yang and Li [68] show that the differential equation $f^3 + \frac{3}{4}f'' = -\frac{1}{4}\sin(3z)$ has exactly three non-constant entire solutions:

$$f_1(z) = \sin(z), \quad f_2(z) = \frac{\sqrt{3}}{2}\cos(z) - \frac{1}{2}\sin(z), \quad f_3(z) = -\frac{\sqrt{3}}{2}\cos(z) - \frac{1}{2}\sin(z).$$

Later on, it is shown in [67] that the equation

$$f^3 + p(z)f'' = c\sin(bz), \tag{4.2}$$

where b and c are non-zero complex numbers and $p(z)$ is a polynomial, does not admit transcendental entire solutions, unless $p(z) = p$ is a constant. In this case, equation (4.2) possesses three distinct transcendental entire solutions, provided that $(pb^2/27)^3 = \frac{1}{4}c^2$.

The right-hand side of (4.2) can be written as a linear combination of e^{ibz} and e^{-ibz}. Based on this observation, Li and Yang [44] consider a nonlinear differential equation of the form

$$f^3 + af'' = p_1 e^{\lambda z} + p_2 e^{-\lambda z}, \tag{4.3}$$

where a, p_1, p_2 and λ are non-zero constants, and they show that (4.3) has transcendental entire solutions if and only if $p_1 p_2 + (a\lambda^2/27)^3 = 0$. Moreover, they prove that f has only three possible forms:

$$f(z) = \rho_j e^{\lambda z/3} - \left(\frac{a\lambda^2}{27\rho_j}\right) e^{-\lambda z/3}, \quad j = 1, 2, 3,$$

where ρ_j, $j = 1, 2, 3$, are the three cubic roots of p_1.

In [43], the equation (4.3) is generalized to

$$f^n + P(z, f) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}, \quad n \geq 3, \tag{4.4}$$

where $P(z, f)$ is a differential polynomial in f and its derivatives of degree at most $n-2$, $p_1, p_2, \alpha_1, \alpha_2$ are non-zero constants and $\alpha_1 \neq \alpha_2$. In fact, it is proved that if f is a transcendental meromorphic solution of (4.4) with few poles in the sense that $N(r, f) = S(r, f)$, then f has only three possible forms:

$$f(z) = c_0(z) + c_1 e^{\alpha_1 z/n}, \quad f(z) = c_0(z) + c_2 e^{\alpha_2 z/n}, \quad f(z) = c_1 e^{\alpha_1 z} + c_2 e^{\alpha_2 z},$$

provided that $\alpha_1 + \alpha_2 = 0$, where $c_0(z)$ is a small function of f and c_1, c_2 are constants satisfying $c_1^n = p_1$, $c_2^n = p_2$.

The results for the equation (4.4) are extended to equations of the form

$$f^n + P(z, f) = p_1(z) e^{\alpha_1(z)} + p_2(z) e^{\alpha_2(z)}, \quad n \geq 3, \tag{4.5}$$

where $p_1(z)$, $p_2(z)$ are rational functions and $\alpha_1(z)$, $\alpha_2(z)$ are non-constant polynomials [46]. In particular, the following theorem reveals the form of the meromorphic solutions of (4.5).

Theorem 4.2 ([46]). *Let $n \geq 3$, and let $P_d(z, f)$ be a differential polynomial in f and its derivatives of degree d with rational functions as its coefficients. Suppose that p_1, p_2 are rational functions and α_1, α_2 are polynomials. If $d \leq n-2$, the equation* (4.5) *admits a meromorphic function f with finitely many poles. Then α_1'/α_2' is a rational number. Furthermore, only one of the following four cases holds:*

(1) *$f(z) = q(z)e^{P(z)}$ and $\frac{\alpha_1'}{\alpha_2'} = 1$, where $q(z)$ is a rational function and $P(z)$ is a polynomial with $nP'(z) = \alpha_1' = \alpha_2'$;*

(2) *$f(z) = q(z)e^{P(z)}$ and either $\frac{\alpha_1'}{\alpha_2'} = \frac{k}{n}$ or $\frac{\alpha_1'}{\alpha_2'} = \frac{n}{k}$, where $q(z)$ is a rational function, k is an integer with $1 \leq k \leq d$ and $P(z)$ is a polynomial with $nP'(z) = \alpha_1'$ or $nP'(z) = \alpha_2'$;*

(3) *f satisfies the first order linear differential equation* $f' = \frac{1}{n}\left(\frac{p_2'}{p_2} + \alpha_2'\right) f + \psi$ *and* $\frac{\alpha_1'}{\alpha_2'} = \frac{n-1}{n}$ *or f satisfies the first order linear differential equation* $f' = \frac{1}{n}\left(\frac{p_1'}{p_1} + \alpha_1'\right) f + \psi$ *and* $\frac{\alpha_1'}{\alpha_2'} = \frac{n}{n-1}$*, where ψ is a rational function;*

(4) $f(z) = \gamma_1(z)e^{\beta(z)} + \gamma_2(z)e^{-\beta(z)}$ *and* $\frac{\alpha_1'}{\alpha_2'} = -1$*, where γ_1, γ_2 are rational functions and $\beta(z)$ is a polynomial with $n\beta'(z) = \alpha_1'$ or $n\beta'(z) = \alpha_2'$.*

Other extensions regarding equation (4.5) have appeared recently, e.g., [45,54].

5 Summary of papers

In the following summaries, the notation used in the original papers have been changed to correspond to the previous sections.

5.1 SUMMARY OF PAPER I

In this paper, we provide some results concerning the transcendental meromorphic solutions of the nonlinear differential equation (4.1). In particular, we give improvements for some earlier results such as Theorem 4.2. As discussed in Chapter 4, the previous results regarding the equation (4.1) concern the case when $h(z)$ has the particular form

$$h(z) = p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)}, \tag{5.1}$$

where $p_1(z), p_2(z)$ are small functions of f and $\alpha_1(z), \alpha_2(z)$ are entire functions. The main idea in Paper I is to study the equation (4.1) in the case when $h(z)$ is a meromorphic solution of a linear differential equation

$$h'' + r_1(z)h' + r_0(z)h = r_2(z), \tag{5.2}$$

where $r_0(z), r_1(z)$ and $r_2(z)$ are rational functions. In this case, the order of $h(z)$ can be a half-integer, $h(z)$ can be a rational function, and it can be a special function such as the Airy integral. Meanwhile, the order of $h(z)$ in the case of (5.1) is either an integer or equals infinity. The case when $h(z)$ satisfies (5.2) is new and has not been covered in previous studies. Note that each function $h(z)$ of the form (5.1), where $p_1(z), p_2(z)$ are rationals and $\alpha_1(z), \alpha_2(z)$ are polynomials, is a solution of an equation of the form (5.2).

We begin with a general result, Theorem 5.1 below, which asserts that if f is a transcendental meromorphic solution of (4.1), then either f has finitely many zeros and poles or the number of zeros and poles of f dominates the growth of $T(r, f)$ in a certain way. In addition to giving properties of solutions, Theorem 5.1 is considered as an extension of the Tumura-Clunie theorem. Here and henceforth, the total degree of $P(z, f)$ as a polynomial in f and its derivatives is denoted by γ_P.

Theorem 5.1. *Let $n \geq 2$, $\gamma_P \leq n-1$, and let f and h be meromorphic solutions of* (4.1) *and* (5.2), *respectively, and assume that f is transcendental. Then one of the following holds:*

(1) *$\rho(h)$ is an integer, f is of the form $f(z) = q(z)e^{\alpha(z)}$, where q is a rational function, α is a non-constant polynomial, and*

$$T(r,h) = nT(r,f) + S(r,f).$$

Furthermore, if r_1 and r_0 are polynomials, then q is a constant.

(2) *f satisfies*

$$T(r,f) \leq \frac{2}{n-\gamma_P}\overline{N}\left(r,\frac{1}{f}\right) + N(r,f) + \frac{2}{n-\gamma_P}\overline{N}(r,f) + S(r,f).$$

Theorem 5.1 can be expressed in terms of the deficiencies (2.8) and (2.9).

Corollary 5.1. *Under the assumptions of Theorem 5.1, if*

$$\Theta(0,f) + \frac{n-\gamma_P}{2}\delta(\infty,f) + \Theta(\infty,f) > 2, \tag{5.3}$$

then the conclusion of Theorem 5.1(1) holds.

We offer the following example, which illustrates Theorem 5.1 and Corollary 5.1, and shows the sharpness of the inequality (5.3).

Example 5.1. (1) The meromorphic function $f(z) = e^{2z}/(e^z - 1)$ has no zeros and satisfies

$$T(r,f) = 2r/\pi + O(1), \quad \overline{N}(r,f) = N(r,f) = r/\pi + O(1).$$

Thus $\Theta(\infty,f) = \delta(\infty,f) = 1/2$ and $\Theta(0,f) = 1$. Moreover, f solves the equation

$$f^3 - \frac{1}{2}f'' + \frac{9}{2}f' - 10f = e^{3z} + 3e^{2z},$$

where the function $h(z) = e^{3z} + 3e^{2z}$ solves the equation $h'' - 4h' + 3h = 0$. We have $\Theta(0,f) + \frac{3-1}{2}\delta(\infty,f) + \Theta(\infty,f) = 2$.

(2) The entire function $f(z) = e^{z/4} + e^{-z/4}$ satisfies

$$T(r,f) = \frac{r}{2\pi} + O(1), \quad \overline{N}(r,1/f) = N(r,1/f) = \frac{r}{2\pi} + O(1).$$

Thus $\Theta(0,f) = \delta(0,f) = 0$ and $\Theta(\infty,f) = \delta(\infty,f) = 1$. Moreover, f solves the equation

$$f^4 - 64ff'' + 2 = e^z + e^{-z},$$

and we have $\Theta(0,f) + +\frac{4-2}{2}\delta(\infty,f) + \Theta(\infty,f) = 2$.

In Theorem 5.2 below, we consider meromorphic solutions f of (4.1) with few poles in the sense that $N(r,f) = S(r,f)$. We give a more precise estimate for the growth of such solutions f when the coefficients of $P(z,f)$ are rational functions. We use the notation

$$r_0(z) \sim C_0 z^m \quad \text{and} \quad r_1(z) \sim C_1 z^l, \tag{5.4}$$

as $z \to \infty$, where $r_0(z)$ and $r_1(z)$ are the coefficients in (5.2), $C_0, C_1 \in \mathbb{C}$, $C_0 \neq 0$ and $m, l \in \mathbb{Z}$. The notation $N_{1)}(r,1/f)$ stands for the integrated counting function of simple zeros of the function f.

Theorem 5.2. *Let $n \geq 3$, and let f be a transcendental meromorphic solution of* (4.1), *where h is a transcendental meromorphic solution of* (5.2), *$P(z,f)$ has rational coefficients, and $\gamma_P \leq n-2$. If $N(r,f) = S(r,f)$, then f has finitely many poles, $\rho(h) = \rho(f)$, and one of the following holds:*

(1) *The conclusion of Theorem 5.1(1) holds.*

(2) *$T(r,f) = N_{1)}(r,1/f) + O(\log r)$, the function f is of order $1 + m/2$, and one of the following two situations for the parameters in* (5.4) *occur.*

(i) *We have* $l \leq -1 \leq m$. *Moreover,*

$$\log M(r, f) \sim \frac{2\sqrt{|C_0|}}{n(m+2)} r^{1+m/2}, \quad r \to \infty.$$

(ii) *We have* $m = 2l \geq 0$ *and* $C_0 = \frac{n(n-1)}{(2n-1)^2} C_1^2$. *Moreover,*

$$\log M(r, f) \sim \frac{2\sqrt{|C_0|}}{\sqrt{n(n-1)}(m+2)} r^{1+m/2}, \quad r \to \infty.$$

Remark 5.1. From the proof of Theorem 5.2, it follows in the sub-case (i) that f satisfies a second-order differential equation

$$f'' + R(z)f' + S(z)f = 0,$$

where $R(z)$ and $S(z)$ are rational functions such that $|R(z)| \asymp |r_1(z)|$ and $|S(z)| \sim |r_0(z)|/n^2$, as $z \to \infty$. In the sub-case (ii), f satisfies a first order differential equation

$$f' + S(z)f = Q(z),$$

where $S(z)$ and $Q(z)$ are non-zero rational functions and $|S(z)| \sim |r_1(z)|/(2n-1)$, as $z \to \infty$.

We give examples to show that the results in Paper I are different from those in previous works (Section 4). We emphasize the cases when h has half-integer order and when h is a rational function.

Example 5.2. The function $f(z) = 2\cos\frac{\sqrt{z}}{3}$ of order $1/2$ satisfies the nonlinear differential equation

$$f^3 + 108zf'' + 54f' = 2\cos\sqrt{z}, \tag{5.5}$$

where $h(z) = 2\cos\sqrt{z}$ is an entire solution of the equation

$$h'' + \frac{1}{2z}h' + \frac{1}{4z}h = 0.$$

This example underlies the sub-case (i) in Theorem 5.2. According to Remark 5.1, here f satisfies the linear differential equation

$$f'' + \frac{1}{2z}f' - \frac{1}{36z}f = 0.$$

Examples of solutions with any pregiven half-integer orders are given in [27, Example 3.5].

Example 5.3. The meromorphic function $f(z) = \frac{1}{e^z-1} + z$ solves the nonlinear differential equation

$$f^3 - \frac{1}{2}f'' + \left(3z - \frac{3}{2}\right)f' - (3z^2 - 3z + 1)f = -2z^3 + 3z^2 + 2z - \frac{3}{2},$$

where $h(z) = -2z^3 + 3z^2 + 2z - \frac{3}{2}$ is a rational solution of the linear differential equation

$$h'' - \frac{z}{3}h' + h = z^2 - \frac{32}{3}z + \frac{9}{2}.$$

Lemma 5.1 below on linear differential equations plays an important role in proving Theorem 5.2, and gives a concrete list of possible orders as well as possible maximum modulus types for f, i.e., the possible values for the constant C in (3.3). The possible types of solutions of higher order differential equation (3.1) polynomial coefficients are not known.

Lemma 5.1. *Let f be a transcendental meromorphic solution of*

$$f'' + R(z)f' + S(z)f = T(z), \tag{5.6}$$

where the coefficients $R(z) \not\equiv 0, S(z) \not\equiv 0, T(z)$ are rational functions such that $R(z) \sim C_R z^n$ and $S(z) \sim C_S z^m$ as $z \to \infty$, where $C_R, C_S \in \mathbb{C}$ and $n, m \in \mathbb{Z}$. Then f has at most finitely many poles and

$$\log M(r, f) \sim C r^{\rho}, \quad r \to \infty,$$

with the following possibilities for C and ρ:

(1) *If $m > 2n$, then $\rho = 1 + \frac{m}{2} \geq 1/2$ and $C = \frac{2\sqrt{|C_S|}}{m+2}$.*

(2) *If $n \leq m < 2n$, then we have two possibilities:*

  (i) *$\rho = n + 1 \geq 1$ and $C = \frac{|C_R|}{1+n}$,*

  (ii) *$\rho = 1 + m - n \geq 1$ and $C = \frac{|C_S|}{(1+m-n)|C_R|}$.*

(3) *If $m < n$, then $\rho = 1 + n \geq 1$ and $C = \frac{|C_R|}{1+n}$.*

(4) *If $m = 2n$, then $\rho = 1 + n \geq 1$ and $C = \frac{|X|}{1+n}$, where X is a complex solution of the quadratic equation $X^2 + C_R X + C_S = 0$.*

If $R(z) \equiv 0$, then only Case (1) is possible. In all cases, $\rho \geq 1/2$.

Next we state a consequence of Theorem 5.2, which treats the equation (4.1) in the case when h has the form (5.1).

Corollary 5.2. *Let $n \geq 3$, and let f be a transcendental meromorphic solution of (4.1), where $P(z, f)$ has rational coefficients, $\gamma_P \leq n - 2$, and h is of the form (5.1), where p_1, p_2 are rational functions such that $p_1 p_2 \not\equiv 0$, and α_1, α_2 are non-constant polynomials normalized such that $\alpha_1(0) = 0 = \alpha_2(0)$. Write*

$$\alpha_j(z) = a_j z^{s_j} + O\left(z^{s_j - 1}\right), \quad j = 1, 2.$$

If $N(r, f) = S(r, f)$, then f has finitely many poles, $\rho(f) = \rho(h) = s_1 = s_2$, and f takes one of the following forms:

(1) *$f(z) = q(z)e^{\alpha(z)}$, where q is a non-zero rational function and α is a non-constant polynomial normalized with $\alpha(0) = 0$. Moreover, the following conclusions hold.*

  (i) *If $a_1 = a_2$, then $n\alpha = \alpha_1 = \alpha_2$, $q^n = p_1 + p_2$ and $P(z, f) \equiv 0$. In particular, if p_1, p_2 are polynomials, then q is also a polynomial.*

  (ii) *The case when $|a_1| = |a_2|$ and $a_1 \neq a_2$ is not possible.*

  (iii) *If $|a_1| \neq |a_2|$, say $|a_1| > |a_2|$, then $n\alpha = \alpha_1$, $q^n = p_1$ and $P(z, f) \equiv h_2$.*

(2) $f(z) = q_1(z)e^{\beta(z)} + q_2(z)e^{-\beta(z)}$, *where q_1, q_2 are non-zero rational functions and β is a non-constant polynomial such that $n\beta = \pm\alpha_1$ and $\alpha_1 = -\alpha_2$.*

(3) $f(z) = q_1(z)e^{\frac{\alpha_1(z)+\alpha_2(z)}{2n-1}} + q_2(z)$, *where q_1, q_2 are non-zero rational functions and* $\max\{|a_1|, |a_2|\}/\min\{|a_1|, |a_2|\} = n/(n-1)$.

Corollary 5.2 is an improvement of Theorem 4.2. For example, the assumption "f has finitely many poles" in Theorem 4.2 can be replaced with the much less restrictive assumption "$N(r, f) = S(r, f)$".

5.2 SUMMARY OF PAPER II

In Paper II, we prove some new results regarding the growth of solutions of (3.1), and give several refinements of Frei's theorem (Theorem 3.3) and its unit disc counterparts (Theorems 3.6 and 3.7). Paper II also discusses analogous results for difference and q-difference equations, but we do not mention these results here due to the subject of our dissertation.

The key idea in Frei's theorem is that the growth of the coefficient $A_p(z)$ dominates the growth of the rest of the coefficients $A_{p+1}(z), \ldots, A_{n+1}(z)$. Based on this idea, many improvements of Frei's theorem have appeared in the literature, in which the dominance of $A_p(z)$ is expressed via a growth scale (order, iterated order, $[k, j]$-order, ...); see for example [25, 36, 48, 60] and the references therein. Chyzhykov and Semochko show in [9] by means of an example that the aforementioned growth scales have the disadvantage of not covering functions of arbitrary growth [9, Example 1.4]. For that, they introduced a more general growth scale, which does not have the disadvantage of the previous scales, and depends on an auxiliary real function satisfying certain conditions.

The results in Paper II are different from the existing improvements of Frei's theorem in the sense that we do not rely on any growth scale. Instead, we compare the growth of solutions directly to the growth of the dominant coefficient $A_p(z)$. All the previous improvements cited above follow as special cases of our results.

5.2.1 Complex plane

In this section, we offer three ways to express the dominance of the transcendental coefficient $A_p(z)$, and in particular, we obtain the lower bound for the number of solutions f of (3.1) that satisfy

$$\log T(r, f) \gtrsim T(r, A_p) \tag{5.7}$$

outside an exceptional set. From (2.2), we see that solutions f satisfying (5.7) are of infinite order. Thus Frei's theorem follows as a special case. The solutions f satisfying (5.7) are called *rapid solutions*. We show in the first two results in this section that all rapid solutions are admissible solutions in the sense of (3.4). Hence, we deduce from Theorem 3.4 that the value 0 is the only possible finite deficient value for all the rapid solutions f of (3.1).

The first result is devoted to expressing the dominance of $A_p(z)$ by means of the Nevanlinna characteristic function.

Theorem 5.3. *Let the coefficients $A_0(z), \dots, A_{n-1}(z)$ in* (3.1) *be entire functions such that at least one of them is transcendental. Suppose that $p \in \{0, \dots, n-1\}$ is the smallest index such that*

$$\limsup_{r\to\infty} \sum_{j=p+1}^{n-1} \frac{T(r, A_j)}{T(r, A_p)} < 1. \tag{5.8}$$

Then $A_p(z)$ is transcendental, and every solution base of (3.1) *has at least $n-p$ rapid solutions f for which*

$$T(r, A_p) \lesssim \log T(r, f) \lesssim \frac{R+r}{R-r} T(R, A_p), \quad r \notin E, \tag{5.9}$$

where $E \subset [0, \infty)$ has finite linear measure, and $r < R < \infty$. For these solutions, the value 0 *is the only possible finite deficient value.*

When $p = n-1$, the sum in (5.8) will be considered to be zero, and the same situation applies in any sum similar to the one in (5.8).

The dominance of $A_p(z)$ in the sense of (5.8) has already been introduced in [23, Theorem 5.6]. The conclusion in [23, Theorem 5.6] addresses the number of solutions f that have slow growth in the sense that

$$\log T(r, f) = o(T(r, A_p)), \quad r \to \infty,\ r \notin E,$$

where $E \subset [0, \infty)$ is a set of finite linear measure. However, solutions with slow growth may grow significantly slower than any of the coefficients [28, Example 2.1].

Comparing (5.9) with (2.1), one may expect that $\log T(r, f)$ and $\log M(r, A_p)$ are asymptotically comparable and, in fact, is what we prove in the second result, where we express the dominance of $A_p(z)$ by means of maximum modulus.

Theorem 5.4. *Let the coefficients $A_0(z), \dots, A_{n-1}(z)$ in* (3.1) *be entire functions such that at least one of them is transcendental. Suppose that $p \in \{0, \dots, n-1\}$ is the smallest index such that*

$$\limsup_{r\to\infty} \sum_{j=p+1}^{n-1} \frac{\log^+ M(r, A_j)}{\log^+ M(r, A_p)} < 1. \tag{5.10}$$

Then $A_p(z)$ is transcendental, and every solution base of (3.1) *has at least $n-p$ rapid solutions f for which*

$$\log T(r, f) \asymp \log M(r, A_p), \quad r \notin E, \tag{5.11}$$

where $E \subset [0, \infty)$ has finite linear measure. For these solutions, the value 0 *is the only possible finite deficient value.*

Before stating the third result in this section, we discuss the Theorems 5.3 and 5.4. We notice that neither Theorem 5.3 nor Theorem 5.4 is stronger than the other regarding the number of rapid/admissible solutions. However, we see that Theorem 5.4 is stronger in its conclusion, as it gives a specific relationship between rapid/admissible solutions and the coefficients. We will show this by means of examples. For the sake of simplicity, all the examples from this point will concern the second order linear differential equation

$$f'' + A_1(z) f' + A_0(z) f = 0. \tag{5.12}$$

The following example shows that Theorem 5.3 is sometimes stronger than Theorem 5.4 regarding the number of rapid solutions.

Example 5.4. Let $A_1(z) = e^{e^z}$; hence from [22, p. 7] we have

$$T(r, A_1) \asymp \frac{e^r}{\sqrt{r}} \quad \text{and} \quad \log M(r, A_1) = e^r.$$

Let $A_0(z)$ be an entire function satisfying

$$T(r, A_0) \sim \log M(r, A_0) \sim 2T(r, A_1), \quad r \to \infty.$$

Such a function $A_0(z)$ exists by Theorem 2.2. Therefore,

$$\limsup_{r\to\infty} \frac{T(r, A_1)}{T(r, A_0)} = \frac{1}{2} < 1.$$

By Theorem 5.3, every non-trivial solution f of (5.12) satisfies

$$\frac{e^r}{\sqrt{r}} \asymp T(r, A_0) \lesssim \log T(r, f).$$

In contrast, we have

$$\limsup_{r\to\infty} \frac{\log M(r, A_1)}{\log M(r, A_0)} = \limsup_{r\to\infty} \frac{\log M(r, A_1)}{2T(r, A_1)} = \infty,$$

which means that $p = 1$ for Theorem 5.4. Thus, by Theorem 5.4, every solution base of (5.12) has at least one solution f_0 satisfying $\log T(r, f_0) \asymp \log M(r, A_1) = e^r$.

Note that the upper bound of $\log T(r, f)$ in (5.9) cannot be reduced to $T(r, A_p)$. Indeed, this is the case in Example 5.4 above, where the asymptotic inequality $\log T(r, f) \lesssim \frac{e^r}{\sqrt{r}}$ does not hold for all rapid solutions f since there exists a rapid solution f_0 satisfying $\log T(r, f_0) \asymp e^r$.

In the next example, we show that Theorem 5.4 could be stronger than Theorem 5.3 regarding the number of rapid solutions.

Example 5.5. Let $A_0(z) = E_{1/\varrho}(z)$ be Mittag-Leffler's function of order $\varrho > 1/2$. We have, from [22, p. 19],

$$T(r, A_0) \sim \frac{1}{\pi\varrho} \log M(r, A_0) \sim \frac{1}{\pi\varrho} r^{\varrho}, \quad r \to \infty.$$

Let $A_1(z)$ be an entire function satisfying

$$T(r, A_1) \sim \log M(r, A_1) \sim T(r, A_0), \quad r \to \infty.$$

Therefore,

$$\limsup_{r\to\infty} \frac{T(r, A_1)}{T(r, A_0)} = 1$$

and

$$\limsup_{r\to\infty} \frac{\log M(r, A_1)}{\log M(r, A_0)} = \limsup_{r\to\infty} \frac{T(r, A_0)}{\log M(r, A_0)} = \frac{1}{\pi\varrho} < 1.$$

Thus, Theorem 5.4 is stronger than Theorem 5.3 in this case.

In the third result, it is possible to detect the number of rapid solutions when $A_p(z)$ dominates the rest of the coefficients along a *maximum curve* of $A_p(z)$, i.e., a curve belongs to $\mathcal{M}_{A_p} := \{z \in \mathbb{C} : |A_p(z)| = M(|z|, A_p)\}$; see, e.g., [62].

Theorem 5.5. *Let the coefficients $A_0(z), \ldots, A_{n-1}(z)$ in* (3.1) *be entire functions. Suppose that $p \in \{0, \ldots, n-1\}$ is the smallest index such that $A_p(z)$ is transcendental and*

$$\limsup_{\substack{z\to\infty \\ z\in\Gamma}} \sum_{j=p+1}^{n-1} \frac{1}{\eta_j} \frac{|A_j(z)|^{\eta_j}}{|A_p(z)|} < 1 \tag{5.13}$$

holds for some constants $\eta_j > 1$, where Γ is a maximum curve for $A_p(z)$. Then every solution base of (3.1) *has at least $n-p$ rapid solutions f for which*

$$\log T(r, f) \gtrsim \log M(r, A_p), \quad r \notin E,$$

where $E \subset [0, \infty)$ has finite linear measure.

The rapid solutions in Theorem 5.5 are not admissible solutions due the local restriction of the growth imposed on the coefficients by (5.13). In addition, the asymptotic comparability between $\log T(r, f)$ and $\log M(r, A_p)$ does not always occur as shown in the following example.

Example 5.6. All non-trivial solutions f of the equation $f'' + e^{-z^2} f' + e^z f = 0$ satisfy $\log T(r, f) \gtrsim \log M(r, e^z) = r$ since the condition (5.13) holds for $p = 0$ along the positive real axis, which is the maximum curve for e^z. However, the asymptotic inequality $\log T(r, f) \lesssim \log M(r, e^z)$ does not hold for all solutions, because there exists a solution f_0 satisfying $\log T(r, f_0) \asymp \log M(r, e^{-z^2}) = r^2$ by Theorem 5.4.

Example 5.6 also shows that Theorem 5.5 can be stronger than Theorems 5.3 and 5.4 regarding the number of infinite order solutions.

The next example shows that we can use the main results above to find extra number of infinite order solutions.

Example 5.7. The function $f_1(z) = e^{e^z}$ is an infinite order solution of the equation

$$f'' + (e^{z^2} - e^z) f' - (e^{z^2+z} + e^z) f = 0. \tag{5.14}$$

Let f_2 be any solution of (5.14) linearly independent of f_1. Frei's theorem cannot be used to conclude that f_2 is of infinite order. However, according to any of Theorems 5.3, 5.4 or 5.5, f_2 must satisfy $\log T(r, f_2) \gtrsim \log M(r, e^{z^2}) \asymp T(r, e^{z^2}) \asymp r^2$. Meanwhile, $\log T(r, f_1) \asymp r$.

5.2.2 Unit disc

In this section, we introduce five different ways to express the dominance of the analytic coefficient $A_p(z)$, and, analogously to the complex plane situation, we consider the number of rapid solutions which are slightly different from the rapid situations in $\mathbb{C}$. Here, the rapid solutions of (3.1) in $\mathbb{D}$ are considered according to the two formulations of Frei's theorem in Theorems 3.6 and 3.7. Hence, the following two types of solutions of (3.1) with coefficients analytic in $\mathbb{D}$ are considered as rapid solutions:

(T1) Solutions f satisfying (5.7), where $A_p(z)$ is admissible function.

(T2) Solutions f satisfying

$$\log T(r,f) \gtrsim \log \int_{D(0,r)} |A_p(z)|^{\frac{1}{n-p}} dm(z),$$

where $A_p(z) \notin \mathcal{A}^{-\infty}$. Here, $dm(z)$ is the Lebesgue measure in the disc $D(0,r)$.

The following two results are unit disc counterparts of Theorems 5.3 and 5.4.

Theorem 5.6. *Let the coefficients $A_0(z), \ldots, A_{n-1}(z)$ in* (3.1) *be analytic functions in $\mathbb{D}$ such that at least one of them is admissible. Suppose that $p \in \{0, \ldots, n-1\}$ is the smallest index such that*

$$\limsup_{r\to 1^-} \sum_{j=p+1}^{n-1} \frac{T(r,A_j)}{T(r,A_p)} < 1. \tag{5.15}$$

Then $A_p(z)$ is admissible function, and every solution base of (3.1) *has at least $n-p$ rapid solutions f for which*

$$T(r,A_p) \lesssim \log T(r,f) \lesssim \frac{R+r}{R-r} T(R,A_p), \quad r \notin E, \tag{5.16}$$

where $E \subset [0,1)$ is a set with $\int_E \frac{dr}{1-r} < \infty$, and $0 < r < R < 1$. For these solutions, the value 0 is the only possible finite deficient value.

Theorem 5.7. *Let the coefficients $A_0(z), \ldots, A_{n-1}(z)$ in* (3.1) *be analytic in $\mathbb{D}$. Suppose that $p \in \{0, \ldots, n-1\}$ is the smallest index such that $A_p(z)$ is admissible function and*

$$\limsup_{r\to 1^-} \sum_{j=p+1}^{n-1} \frac{\log^+ M(r,A_j)}{\log^+ M(r,A_p)} < 1. \tag{5.17}$$

Then every solution base of (3.1) *has at least $n-p$ rapid solutions f for which*

$$\log T(r,f) \asymp \log M(r,A_p), \quad r \notin E, \tag{5.18}$$

where $E \subset [0,1)$ is a set with $\int_E \frac{dr}{1-r} < \infty$. For these solutions, the value 0 is the only possible finite deficient value.

From (5.16) or (5.18), using (3.5), each solution base of (3.1) contains at least $n-p$ linearly independent solutions of infinite order. Thus Theorem 3.7 is a particular case of Theorems 5.6 and 5.7.

Note that the statement "$A_p(z)$ is admissible" is a condition instead of a conclusion in Theorem 5.7, unlike in Theorem 5.6. The reason is that the admissibility is defined by means of the Nevanlinna characteristic, and hence the condition (5.17) does not necessarily imply the admissibility of $A_p(z)$.

By using results from [47, Theorem I], we can construct examples analogous to Examples 5.4 and 5.5. Hence, neither Theorem 5.6 nor Theorem 5.7 is stronger than the other regarding the number of rapid solutions [28, Example 3.3].

We now give a unit disc counterpart of Theorem 5.5, where the maximum curve of $A_p(z)$ in $\mathbb{D}$ is defined as a curve emanating from the origin and tending to a point on $\partial\mathbb{D}$ and consists of points $z \in \mathbb{D}$ for which $|A_p(z)| = M(|z|, A_p)$.

Theorem 5.8. *Let the coefficients* $A_0(z), \ldots, A_{n-1}(z)$ *in* (3.1) *be analytic functions in* $\mathbb{D}$. *Suppose that* $p \in \{0, \ldots, n-1\}$ *is the smallest index such that* $A_p(z)$ *is admissible and*

$$\limsup_{\substack{z \to 1^- \\ z \in \Gamma}} \sum_{j=p+1}^{n-1} \frac{1}{\eta_j} \frac{|A_j(z)|^{\eta_j}}{|A_p(z)|} < 1$$

holds for some constants $\eta_j > 1$, *where* Γ *is a maximum curve of* $A_p(z)$. *Then every solution base of* (3.1) *has at least* $n-p$ *rapid solutions* f *for which*

$$\log T(r, f) \gtrsim \log M\left(r, A_p\right), \quad r \notin E,$$

where $E \subset [0,1)$ *is a set with* $\int_E \frac{dr}{1-r} < \infty$.

In the following two results, we use the integral mean to measure the growth of analytic functions in $\mathbb{D}$.

Theorem 5.9. *Let the coefficients* $A_0(z), \ldots, A_{n-1}(z)$ *in* (3.1) *be analytic functions in* $\mathbb{D}$. *Suppose that* $p \in \{0, \ldots, n-1\}$ *is the smallest index such that* $A_p(z)$ *is admissible and*

$$\limsup_{r \to 1^-} \sum_{j=p+1}^{n-1} \binom{n-j}{n-p} \frac{\displaystyle\int_0^{2\pi} |A_j(re^{i\theta})|^{\frac{1}{n-j}} d\theta}{\displaystyle\int_0^{2\pi} |A_p(re^{i\theta})|^{\frac{1}{n-p}} d\theta} < 1. \tag{5.19}$$

Then every solution base of (3.1) *has at least* $n-p$ *solutions* f *for which*

$$\log T(r, f) \asymp \log \int_0^{2\pi} |A_p(re^{i\theta})|^{\frac{1}{n-p}} d\theta, \quad r \notin E,$$

where $E \subset [0,1)$ *is a set with* $\overline{d}(E) < 1$. *These solutions are rapid in the sense of (T1), and the value* 0 *is their only possible finite deficient value.*

From Jensen's inequality

$$\log^+ \int_0^{2\pi} |A_p(re^{i\theta})|^{\frac{1}{n-p}} d\theta \gtrsim m(r, A_p) = T(r, A_p),$$

it is clear that Theorem 3.7 is a particular case of Theorem 5.9.

The previous four results are suitable with Theorem 3.7, and address the rapid solutions of type (T1). The next result is suitable with Theorem 3.6, and addresses the rapid solutions type (T2).

Theorem 5.10. *Let the coefficients* $A_0(z), \ldots, A_{n-1}(z)$ *in* (3.1) *be analytic functions in* $\mathbb{D}$ *such that at least one of them does not belong to* $\mathcal{A}^{-\infty}$. *Suppose that* $p \in \{0, \ldots, n-1\}$ *is the smallest index such that*

$$\limsup_{r \to 1^-} \sum_{j=p+1}^{n-1} \binom{n-j}{n-p} \frac{\displaystyle\int_{D(0,r)} |A_j(z)|^{\frac{1}{n-j}} dm(z)}{\displaystyle\int_{D(0,r)} |A_p(z)|^{\frac{1}{n-p}} dm(z)} < 1. \tag{5.20}$$

Then $A_p(z) \notin \mathcal{A}^{-\infty}$, *and every solution base of* (3.1) *has at least* $n-p$ *solutions* f *for which*

$$\log T(r, f) \asymp \log \int_{D(0,r)} |A_p(z)|^{\frac{1}{n-p}} dm(z), \quad r \notin E, \tag{5.21}$$

where $E \subset [0,1)$ is a set with $\int_E \frac{dr}{1-r} < \infty$. These solutions are rapid in the sense of (T2), and the value 0 is their only possible finite deficient value.

The fact that Theorem 3.6 is a particular case of Theorem 5.10 follows from the next result, which is a slight modification of [26, Example 5.4].

Proposition 5.1. *Suppose that $g(z)$ is an analytic function in $\mathbb{D}$. Then $g \notin \mathcal{A}^{-\infty}$ if and only if for any $\kappa \in (0,1)$,*

$$\limsup_{r\to 1^-} \frac{\log^+ \int_{D(0,r)} |g(z)|^\kappa dm(z)}{-\log(1-r)} = \infty.$$

5.3 SUMMARY OF PAPER III

In Paper III, we give some new findings regarding second order differential equations (3.10) and (3.11). Particularly, we use the concept of the *magnitudes of deviation of a function g with respect to ∞*, where g is either the coefficient $A(z)$ in (3.10) or $a(z)$ in (3.11). These magnitudes are introduced by Petrenko [55], and are defined by

$$\beta^-(\infty, g) := \liminf_{r\to\infty} \frac{\log M(r,g)}{T(r,g)} \quad \text{and} \quad \beta^+(\infty, g) := \limsup_{r\to\infty} \frac{\log M(r,g)}{T(r,g)}.$$

If g is of finite lower order μ, then [55, Theorem 1] shows that

$$1 \le \beta^-(\infty, g) \le \mathcal{B}(\mu), \tag{5.22}$$

where

$$\mathcal{B}(\mu) := \begin{cases} \dfrac{\pi\mu}{\sin(\pi\mu)}, & \text{if} \quad 0 \le \mu < \frac{1}{2}, \\ \pi\mu, & \text{if} \quad \mu \ge \frac{1}{2}. \end{cases}$$

5.3.1 Oscillation of solutions

We prove Theorem 5.11 below, which is a generalization of Theorem 3.11. Recall that the lower logarithmic density $\underline{\text{logdens}}(G)$ of a set $G \subset [1,\infty)$ is defined by

$$\underline{\text{logdens}}(G) := \liminf_{r\to\infty} \frac{1}{\log r} \int_{G\cap[1,r]} \frac{dt}{t}.$$

A set of finite logarithmic measure has zero upper logarithmic density.

Theorem 5.11. *Let $\alpha \in (0,1]$, and let $A(z)$ be a transcendental entire function satisfying*

$$T(r,A) \sim \alpha \log M(r,A) \tag{5.23}$$

as $r \to \infty$ outside a set G with $\underline{\text{logdens}}(G) = \beta < 1$. Suppose further that one of the following holds:

(1) $\rho(A) \notin \mathbb{N}$, (2) $\mu(A) < \rho(A)$, (3) $\rho(A) < \frac{1-\beta}{2(1-\alpha)}$.

If E is a product of two linearly independent solutions of (3.10), *then*

$$\lambda(E) \geq \frac{1-\beta}{2(1-\alpha)}.$$

In particular, if $\alpha = 1$, then $\lambda(E) = \infty$.

This theorem also improves the inequality (3.16) in the case $1 \leq \rho(A) < \frac{1-\beta}{2(1-\alpha)}$. Any of the conditions (1)–(3) is necessary as shown by the following simple example.

Example 5.8 ([39, Theorem 5.22])**.** The equation

$$f'' + \left(e^z - \frac{1}{16}\right) f = 0$$

has two linearly independent solutions f_1 and f_2 such that $\lambda(f_1 f_2) = 0$. Here the coefficient $A(z) = e^z - 1/16$ has order $\rho(A) = 1$ and satisfies (5.23) for $\alpha = 1/\pi$ without an exceptional set.

A concrete example for the assumption (3) in Theorem 5.11 is Mittag-Leffler's function of order $\rho \in \left(\frac{1}{2}, \frac{2+\pi}{2\pi}\right)$, which satisfies (5.23) with $\alpha = \frac{1}{\pi\rho}$ and without an exceptional set [22, p. 19].

The lower bound of $\lambda(E)$ in Theorem 5.11 does not depend on $\rho(A)$ or $\mu(A)$, and we see that whenever α is close enough to 1, $\lambda(E)$ is arbitrarily large, without taking into account the values $\rho(A)$ and $\mu(A)$.

The following corollary is a direct consequence of Theorem 5.11.

Corollary 5.3. *Let $A(z)$ be a transcendental entire function. Suppose that one of (1)–(3) with $\beta = 0$ in Theorem 5.11 holds. If E is a product of two linearly independent solutions of* (3.10), *then*

$$\lambda(E) \geq \frac{\beta^+(\infty, A)}{2(\beta^+(\infty, A) - 1)}.$$

In particular, if $\beta^+(\infty, A) = 1$, then $\lambda(E) = \infty$.

5.3.2 Growth of solutions

Here, we give new conditions on the coefficients of (3.11), forcing the solutions to be of infinite order. We define a quantity

$$\xi(a) := \frac{1}{2\pi} \cdot \text{meas}\left(\left\{\theta \in [0, 2\pi) : \limsup_{r\to\infty} \frac{\log^+ |a(re^{i\theta})|}{\log r} < \infty\right\}\right),$$

where meas (E) stands for the linear measure of a set $E \subset [0, 2\pi)$. Clearly $0 \leq \xi(a) \leq 1$. For example, we see that $\xi(a) = 1$ if $a(z)$ is a polynomial, and $\xi(a) = 0$ if $a(z) = e^z + e^{-z}$. A transcendental entire function $a(z)$ with $\xi(a) = 1$ exists [22, Lemma 4.1]. If $a(z) \not\equiv 0$ is a contour integral solution of

$$w^{(n)} + (-1)^{n+1} b w^{(k)} + (-1)^{n+1} z w = 0, \quad n \geq 2,\ n > k > 0,\ b \in \mathbb{C},$$

then [18, Theorem 3] reveals that $\xi(a) \geq \frac{1}{2\pi} \cdot \frac{n\pi}{n+1}$.

In the following result, we use Petrenko's deviation.

Theorem 5.12. *Let $a(z)$ be an entire function such that $\xi(a) > 0$, and let $b(z)$ be a transcendental entire function satisfying $\beta^-(\infty, b) < \frac{1}{1-\xi(a)}$. Then every non-trivial solution of* (3.11) *is of infinite order.*

An entire function $a(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ is said to have Fabry gaps if $\lim \lambda_n / n = \infty$. It has been shown in [13] that a function $a(z)$ with Fabry gaps satisfies

$$\log L(r, a) \sim \log M(r, a), \quad r \to \infty, \tag{5.24}$$

outside a set of zero upper logarithmic density, where $L(r, a) = \min_{|z|=r} |a(z)|$. Consequently, $a(z)$ satisfies

$$T(r, a) \sim \log M(r, a), \quad r \to \infty,$$

outside a set of zero upper logarithmic density. This leads to the following consequence of Theorem 5.12.

Corollary 5.4. *Let $a(z)$ and $b(z)$ be entire functions. Suppose there exists a sector where $\log^+ |a(z)| \lesssim \log |z|$, and suppose that $b(z)$ is a transcendental functions with Fabry gaps. Then every non-trivial solution of* (3.11) *is of infinite order.*

Corollary 5.4 improves Theorems 1.3 and 1.7 in [50].

Using the $\cos \pi\rho$ -theorem, one can easily see that if $\xi(a) > 0$ and $\mu(b) < 1/2$, then every non-trivial solution of (3.11) is of infinite order. The same conclusion holds if

$$1/2 \leq \mu(b) < \frac{1}{\pi(1 - \xi(a))}. \tag{5.25}$$

This follows by (5.22) and Theorem 5.12. In the following result, the condition (5.25) is weakened to $\mu(b) < \frac{1}{2(1-\xi(a))}$.

Theorem 5.13. *Let $a(z)$ be an entire function such that $\xi(a) > 0$, and let $b(z)$ be a transcendental entire function satisfying $\mu(b) < \frac{1}{2(1-\xi(a))}$. Then every non-trivial solution of* (3.11) *is of infinite order.*

To illustrate this theorem, let $a(z)$ be the Mittag-Leffler's function of order $\rho(a) > 1/2$, and let $b(z)$ be a transcendental entire function with $\mu(b) \neq \rho(a)$. Then $\xi(a) = 1 - \frac{1}{2\rho(A)}$ [22, p. 19], so that either $\mu(b) < \rho(a) = \frac{1}{2(1-\xi(a))}$ or $\rho(b) \geq \mu(b) > \rho(a)$. It follows from Theorem 5.13 and [17, Corollary 1] that every non-trivial solution of (3.11) is of infinite order.

5.4 SUMMARY OF PAPER IV

In this paper we prove results on the growth and zero distribution of solutions of equation (3.10) in the case when $A(z)$ is a non-constant polynomial. The results obtained in this case lie under the theory on the asymptotic integration due to Hille [32, Ch. 7.4]. For the convenience of the reader, we rewrite equation (3.10) as

$$f'' + P(z)f = 0 \tag{5.26}$$

where

$$P(z) = p_n z^n + p_{n-1} z^{n-1} + \cdots + p_0, \quad p_n \neq 0, \ n \geq 1.$$

5.4.1 Observations

It is proved in [32, Ch. 7.4] that all but finitely many zeros of a solution $f \not\equiv 0$ of (5.26) lie in $n+2$ ε-sectors

$$W_j(\varepsilon) = \{z : |\arg(z) - \theta_j| < \varepsilon\}, \quad j = 0, \ldots, n+1, \tag{5.27}$$

around the critical rays $\arg(z) = \theta_j$, where

$$\theta_j = \frac{2\pi j - \arg(p_n)}{n+2}, \quad j = 0, \ldots, n+1, \tag{5.28}$$

with $-\pi \leq \arg(p_n) < \pi$. The value $\varepsilon > 0$ in (5.27) is arbitrarily small. Further, it is claimed in [32, p. 342] that if f has infinitely many zeros in a sector $W_j(\varepsilon)$, then these zeros approach the critical ray $\arg(z) = \theta_j$. This remark has later been corrected by Bank [1] and by Hellerstein-Rossi [30], and they prove that the zeros in fact approach the translate $\arg(z + c) = \theta_j$ of the critical ray, where $c = p_{n-1}/np_n$. Moreover, the distance of the zeros z_k from the ray $\arg(z + c) = \theta_j$ is $O(r_k e_n(r_k))$, as $r_k = |z_k| \to \infty$, where

$$e_n(r) = \begin{cases} r^{-2}, & n > 2, \\ r^{-2}\log r, & n = 2, \\ r^{-3/2}, & n = 1. \end{cases} \tag{5.29}$$

In addition to the result mentioned above about the location of zeros of solutions of (5.26), the following conclusions can also be found in [32, Ch. 7.4]:

(A) On all the rays in an open sector $S(\theta_{j-1}, \theta_j)$ determined by any two consecutive critical rays, each solution $f \not\equiv 0$ of (5.26) either blows up on each ray or decays to zero exponentially on each ray. By this we mean, respectively, that

$$\liminf_{r\to\infty} r^{-(n+2)/2} \log |f(re^{i\theta})| > 0 \quad \text{or} \quad \liminf_{r\to\infty} r^{-(n+2)/2} \log |f(re^{i\theta})|^{-1} > 0,$$

for $\theta \in (\theta_{j-1}, \theta_j)$.

(B) If a sector $W_j(\varepsilon)$ contains infinitely many zeros of f, then the number of zeros of f in $W_j(\varepsilon)$ is asymptotically comparable to $r^{(n+2)/2}$.

These conclusions in (A) and (B) are the result of Hille's theory on asymptotic integration combined with Liouville's transformation [32, pp. 339-340]. However, many readers can find the reading of [32, Ch. 7.4] laborious because general statements of some of the basic results and consequences of the asymptotic integration theory are not clearly and fully stated, and the details that would justify several steps in the proofs are omitted. Accordingly, our purposes in Paper IV are to

> write clearly stated results together with rigorous proofs on the growth and zero distribution of solutions of (5.26) that are obtained from Hille's approach. For the zero distribution, we take into account the above convergence rate $O(r_k e_n(r_k))$ of the zeros $\{z_k\}$ toward the *critical translates* $\arg(z + c) = \theta_j$, where $j = 0, \ldots, n+1$.

The process of studying the equation (5.26) is described in Figure 5.1.

$$\begin{tikzcd}[column sep=huge]
f'' + P(z)f = 0 \arrow[r, leftrightarrow, "\text{change of}", "\text{variable}"'] & g'' + Q(z)g = 0 \arrow[d, leftrightarrow, "\text{Liouville's transformation}"] \\
w'' + w = 0 \arrow[r, leftrightarrow, "\text{asymptotic}", "\text{integration}"'] & w'' + [1 - T(\zeta)]w = 0
\end{tikzcd}$$

Figure 5.1: The connection between equations.

1. *Change of variable*: We use $g(z) = f(\mu z - c)$, where μ is a constant satisfying $\mu^{n+2} = p_n^{-1}$ and $c = p_{n-1}/np_n$, to transform (5.26) into
$$g'' + Q(z)g = 0 \tag{5.30}$$
with a normalized polynomial coefficient
$$Q(z) = \begin{cases} z, & n = 1, \\ z^n + a_{n-2}z^{n-2} + \cdots + a_0, & n \geq 2. \end{cases} \tag{5.31}$$

2. *Liouville's transformation*: This transformation is used to transform (5.30) into *perturbed sine equation*
$$w'' + [1 - T(\zeta)]w = 0, \tag{5.32}$$
where $T(\zeta) = O(\zeta^{-2})$ as $\zeta \to \infty$.

3. *Asymptotic integration*: This theory is used to link (5.32) and the sine equation
$$w'' + w = 0. \tag{5.33}$$

The purpose of these three steps is to make a simple connection between equation (5.26) and (5.33), so that the well-known growth and zero distribution properties of the three types of solutions $e^{iz}, e^{-iz}, \sin(z - z_0)$ of (5.33) can be used to prove the analogous properties of the solutions of (5.26).

5.4.2 Main results

In Theorems 5.14 and 5.15 below we write more precise statements of the aforementioned results concerning the equation (5.26). For this purpose, we use the notation
$$c = \frac{p_{n-1}}{np_n}, \quad q = \frac{n+2}{2}, \quad d = \frac{(p_n)^{1/2}}{q}, \tag{5.34}$$
where $(p_n)^{1/2} = \sqrt{|p_n|}\exp\left(i\frac{\arg(p_n)}{2}\right)$ with $-\pi < \arg(p_n) \leq \pi$. The first result concerns the exponential growth and decay of solutions to make (A) precise.

Theorem 5.14. *Let f be a non-trivial solution of* (5.26). *Then the following statements hold:*

(a) *In any given open sector S between consecutive critical rays, f either (i) blows up exponentially on all the rays* $\arg(z) = \theta$ *in S, or (ii) decays exponentially to zero* *on all the rays* $\arg(z) = \theta$ *in* S. *Specifically, on all the rays* $\arg(z) = \theta$ *in* S, f *is asymptotically comparable to either*

$$E_1(z) = \exp\{idz^q(1+o(1))\} \quad \textit{or} \quad E_2(z) = \exp\{-idz^q(1+o(1))\}.$$

Moreover, in each such sector S, *there exist solutions of* (5.26) *of both types (i) and (ii).*

(b) *In any two adjacent sectors* $S(\theta_{j-1}, \theta_j)$ *and* $S(\theta_j, \theta_{j+1})$ *that border one common critical ray* $\arg(z) = \theta_j$, *there cannot exist a ray in* $S(\theta_{j-1}, \theta_j)$ *and another ray in* $S(\theta_j, \theta_{j+1})$ *such that* f *decays exponentially to zero on both rays. Here,* $\theta_{-1} = \theta_{n+1}$.

In Theorem 5.14(a), any branch cut outside the sector S and any branch for the square roots in the expressions $E_1(z)$ and $E_2(z)$ can be chosen as long as they are the same for both $E_1(z)$ and $E_2(z)$. For some choices, the roles of $E_1(z)$ and $E_2(z)$ will be interchanged.

Instead of the classical ε-sectors in (5.27), we consider the domains

$$\Lambda_j = \{z = re^{i\theta} : r > R,\ |\theta - \theta_j| < Ce_n(r)\}$$

and their translates

$$\Lambda_{j,c} = \{z : z + c \in \Lambda_j\}. \tag{5.35}$$

Here $R > 0$ is large enough, $C = C(n, R) > 0$, and c is defined in (5.34). The second result concerns the distribution of zeros of solutions to make (B) precise.

Theorem 5.15. *Let* f *be a non-trivial solution of* (5.26). *Then all but at most finitely many zeros of* f *lie in the union*

$$\bigcup_{j=0}^{n+1} \Lambda_{j,c}.$$

If f *has infinitely many zeros in* $\Lambda_{j,c}$, *then*

$$n(r, \Lambda_{j,c}, 1/f) = \frac{\sqrt{|p_n|}}{q\pi} r^q (1 + o(1)), \quad r \to \infty, \tag{5.36}$$

$$N(r, \Lambda_{j,c}, 1/f) = \frac{\sqrt{|p_n|}}{q^2\pi} r^q (1 + o(1)), \quad r \to \infty, \tag{5.37}$$

where the counting function $n(r, \Lambda_{j,c}, 1/f)$ *refers to only those zeros in* $\Lambda_{j,c}$ *satisfying* $|z| \le r$ *and* $N(r, \Lambda_{j,c}, 1/f)$ *is the corresponding integrated counting function.*

From this result it is clear that the zeros of f approach the critical translates

$$\arg(z + c) = \theta_j, \quad j = 0, \ldots, n+1,$$

emanating from the point $-c$, see Figure 5.2. It is easy to note that the sector $W_j(\varepsilon)$ enclosing the critical ray $\arg z = \theta_j$ contains the essential part of $\Lambda_{j,c}$, no matter how large $|c|$ is. Therefore, it follows that all but at most finitely many zeros of f are located in the union of the sectors $W_j(\varepsilon)$, as has been previously known.

If f has only finitely many zeros in $W_j(\varepsilon)$, then the critical ray $\arg(z) = \theta_j$ is called a *shortage ray* of f, otherwise it is a *non-shortage ray* of f. The third result reveals the interplay between exponential growth/decay and non-shortage/shortage rays.

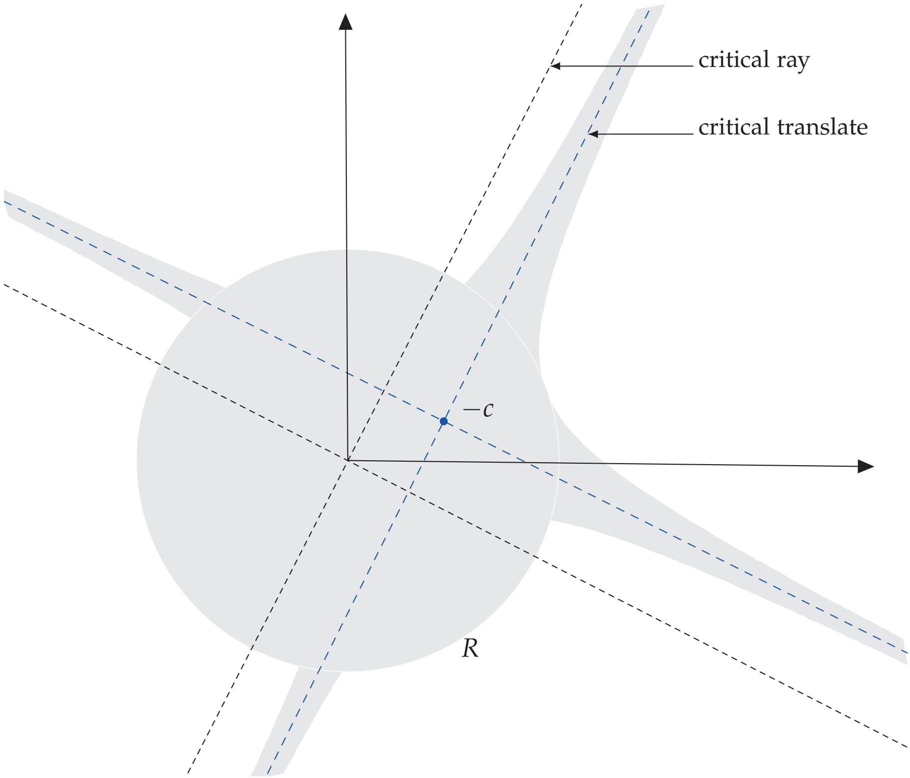

Figure 5.2: Domains $\Lambda_{j,c}$ around the critical translates in the case $n = 2$. All the zeros of f lie in the shaded area when $R > 0$ is large enough.

Theorem 5.16. *Let f be a non-trivial solution of* (5.26). *Then the following statements hold:*

(a) *If f blows up exponentially on each ray in two adjacent sectors $S(\theta_{j-1}, \theta_j)$ and $S(\theta_j, \theta_{j+1})$ that border a common critical ray $\arg(z) = \theta_j$, then the critical ray $\arg(z) = \theta_j$ is non-shortage.*

(b) *If f decays to zero exponentially on all the rays in a sector $S(\theta_j, \theta_{j+1})$, then both critical rays $\arg(z) = \theta_j, \theta_{j+1}$ are shortage.*

5.4.3 Liouville's transformation

We proceed first to give some preliminary concepts. The polynomial $Q(z)$ in (5.31) can be re-written as

$$Q(z) = z^n(1 + \ell(z)), \quad n \geq 1, \tag{5.38}$$

where $\ell(z) \equiv 0$ if $n = 1$ and

$$\ell(z) = \frac{a_{n-2}}{z^2} + \cdots + \frac{a_0}{z^n}, \quad n \geq 2.$$

From (5.28), the critical rays of (5.30) are $\arg(z) = \psi_j$, where

$$\psi_j = \frac{2\pi j}{n+2}, \quad j = 0, 1, \ldots, n+1. \tag{5.39}$$

For $j = 0, \ldots, n+1$, let $G_j(R)$ denote the domain

$$G_j(R) = \left\{z \in \mathbb{C} : |z| > R,\ \psi_{j-1} < \arg(z) < \psi_{j+1}\right\}, \tag{5.40}$$

where $\psi_{-1} = \psi_{n+1} - 2\pi$ and $\psi_{n+2} = \psi_0 + 2\pi$, and $R > 0$ is large enough to satisfy

$$R \geq \max\left\{1, \sqrt{(n-1)M_0}\right\}, \tag{5.41}$$

where $M_0 = 0$ if $n = 1$ and

$$M_0 = \max\{|a_0|, \ldots, |a_{n-2}|\}, \quad n \geq 2.$$

For a non-trivial solution g of (5.30), let g_j denote the restriction of g to $G_j(R)$.

Now, we are ready to define Liouville's transformation. For $z \in G_j(R)$, Liouville's transformation is defined as

$$\begin{cases} \zeta = L_j(z) = \displaystyle\int_{z_0}^{z} \xi^{n/2} \left(1 + \ell\left(\xi\right)\right)^{1/2} d\xi, \\ w_j(\zeta) = z^{n/4} \left(1 + \ell\left(z\right)\right)^{1/4} g_j(z). \end{cases} \tag{5.42}$$

We choose $z_0 = 2Re^{i\psi_j}$. The path of integration in (5.42) is any polygonal path between z_0 and z consisting of at most two line segments lying in $G_j(R)$.

Branches for the square roots and the fourth roots in (5.42) are chosen as follows: For $(1 + \ell(z))^{1/2}$ and $(1 + \ell(z))^{1/4}$, we always use the principal branch

$$-\pi < \arg(1 + \ell(z)) \leq \pi. \tag{5.43}$$

When an expression $w = w(z)$ does not represent $1 + \ell(z)$ and k is a positive integer, $w^{k/2}$ will be defined by $w^{k/2} = (w^{1/2})^k$, where

$$w^{1/2} = |w|^{1/2} \exp\left(i\frac{\arg(w)}{2}\right), \quad \psi_j - \pi < \arg(w) \leq \psi_j + \pi, \tag{5.44}$$

and $w^{k/4}$ will be defined by $w^{k/4} = (w^{1/4})^k$, where

$$w^{1/4} = |w|^{1/4} \exp\left(i\frac{\arg(w)}{4}\right), \quad \psi_j - \pi < \arg(w) \leq \psi_j + \pi. \tag{5.45}$$

Next, we show that $\zeta = L_j(z)$ in (5.42) is well-defined. For $z \in G_j(R)$, where R satisfies (5.41), we have $\ell(z) \equiv 0$ if $n = 1$ and

$$|\ell(z)| \leq \frac{1}{|z|^2} \sum_{s=2}^{n} |a_{n-s}| \leq \frac{(n-1)M_0}{|z|^2} < 1, \quad |z| > R, \quad n \geq 2. \tag{5.46}$$

It follows that $1 + \ell(z)$ lies entirely in the right half-plane for all $|z| > R$. Therefore, by (5.43) it follows that

$$(1 + \ell(z))^{1/2} \quad \text{and} \quad (1 + \ell(z))^{1/4}$$

are analytic in $\{z : |z| > R\}$, and by (5.44) and (5.45) it follows that the functions

$$A(z) = z^{n/2} \left(1 + \ell\left(z\right)\right)^{1/2} \quad \text{and} \quad B(z) = z^{n/4} \left(1 + \ell\left(z\right)\right)^{1/4}$$

are analytic in the domain $\{z : |z| > R,\, \psi_j - \pi < \arg(z) < \psi_j + \pi\}$. In particular, they are analytic in $G_j(R)$, and with the choice of z_0 and the path of integration as above, it follows that $L_j(z)$ is analytic in $G_j(R)$ as well.

The function $w_j(\zeta)$ in (5.42) is well-defined. This follows from the properties of (5.42) provided by the following lemmas that are needed to prove the main results.

Lemma 5.2. *The function $\zeta = L_j(z)$ in* (5.42) *satisfies*

$$\zeta = L_j(z) = \frac{2}{n+2} z^{(n+2)/2} \left(1 + K(z)\right), \quad z \in G_j(R), \tag{5.47}$$

where $|K(z)| = O(e_n(|z|))$ as $z \to \infty$ in $G_j(R)$, where $e_n(r)$ is in (5.29)*. Moreover, we have $|\zeta| \sim \frac{2}{n+2}|z|^{(n+2)/2}$, as $|z| \to \infty$, and*

$$|\arg(1 + K(z))| = O(e_n(|z|)), \quad |z| \to \infty. \tag{5.48}$$

The next lemma shows that $L_j(z)$ is univalent in $G_j(R)$ and maps $G_j(R)$ onto a domain containing

$$\widetilde{G}_j(\delta, \widetilde{R}) = \left\{\zeta \in \mathbb{C} : |\zeta| > \widetilde{R},\ |\arg(\zeta) - \pi j| \le \pi - \delta\right\}, \tag{5.49}$$

for a given small $\delta > 0$. This implies that the function $w_j(\zeta)$ in (5.42) is analytic in a domain containing $\widetilde{G}_j(\delta, \widetilde{R})$.

Lemma 5.3. *For any $j \in \{0, 1, \ldots, n+1\}$, the following two properties hold.*

(1) *The function $L_j(z)$ in* (5.42) *is one-to-one in the domain $G_j(R)$, provided that $R > 0$ is sufficiently large.*

(2) *Let $\delta > 0$ be small enough. Then there exists $\widetilde{R} > 0$ large enough such that $L_j(G_j(R))$ contains $\widetilde{G}_j(\delta, \widetilde{R})$ defined in* (5.49).

Lemma 5.4 below shows that (5.42) transforms equation (5.30) into a perturbed sine equation of the form

$$w_j''(\zeta) + [1 - T(\zeta)]\, w_j(\zeta) = 0, \tag{5.50}$$

where $T(\zeta) = O(\zeta^{-2})$ as $\zeta \to \infty$. The result is briefly stated in [33, p. 180].

Lemma 5.4. *Let $g(z) \not\equiv 0$ be a solution of* (5.30)*, and let $g_j(z)$ be its restriction to $G_j(R)$. Then $w_j(\zeta)$ defined in* (5.42) *satisfies an equation of the form* (5.50)*, where $T(\zeta)$ is analytic in a domain containing the region $\widetilde{G}_j(\delta, \widetilde{R})$, and*

$$T(\zeta) = \frac{1}{4}\left(\frac{Q''(z)}{Q(z)^2} - \frac{5}{4}\frac{Q'(z)^2}{Q(z)^3}\right) = O\left(\frac{1}{\zeta^2}\right), \quad \zeta \to \infty.$$

For the location of the zeros indicated in Theorem 5.15, the following geometric property of $L_j(z)$ is needed.

Lemma 5.5. *Let $\delta, R, \widetilde{R}$ be as in Lemma 5.3, and let $v_0 \in \mathbb{R}$. For $j \in \{0, \ldots, n+1\}$, let $\ell_j : \zeta = (-1)^j u + i v_0$ denote a horizontal half-line in $\widetilde{G}_j(\delta, \widetilde{R})$, where*

$$\begin{cases} u \ge 0, & \text{if} \quad |v_0| > \widetilde{R}, \\ u > \sqrt{\widetilde{R}^2 - v_0^2}, & \text{if} \quad |v_0| \le \widetilde{R}. \end{cases}$$

Then there exists a constant $C = C(n, |v_0|, R) > 0$ such that the pre-image $\mathcal{L}_j$ of ℓ_j under $\zeta = L_j(z)$ is a curve lying in the domain

$$\Lambda_j^* = \{z = re^{i\theta} : |\theta - \psi_j| < Ce_n(r), r > R\}.$$

5.4.4 Asymptotic integration theory

Here, we present the theory of asymptotic integration. The framework is different from that in [32, Chapter 7.4]. Corresponding to (5.50), we consider a more general perturbed sine equation of the form

$$w'' + (1 - F(z))w = 0, \tag{5.51}$$

where $F(z)$ satisfies Hypothesis $\mathbf{F}^+$ below.

Hypothesis $\mathbf{F}^+$. *The function $F(z)$ is analytic in a domain*

$$D^+(\delta_0, R_0) = \{z \in \mathbb{C} : |z| > R_0, |\arg(z)| < \pi - \delta_0\},$$

where $\delta_0 \in (0, \pi)$ and $R_0 > 0$. For each $z \in D^+(\delta_0, R)$, where $R \geq R_0/\sin(\delta_0)$, the integral $\int_z^\infty |F(t)||dt|$ exists along the path of integration given by $t = z + r$, $0 \leq r < \infty$. Moreover, there exists a δ satisfying $\delta > \delta_0$ such that

$$\lim_{R\to\infty} \sup_{z\in D^+(\delta,R)} \int_z^\infty |F(t)||dt| = 0. \tag{5.52}$$

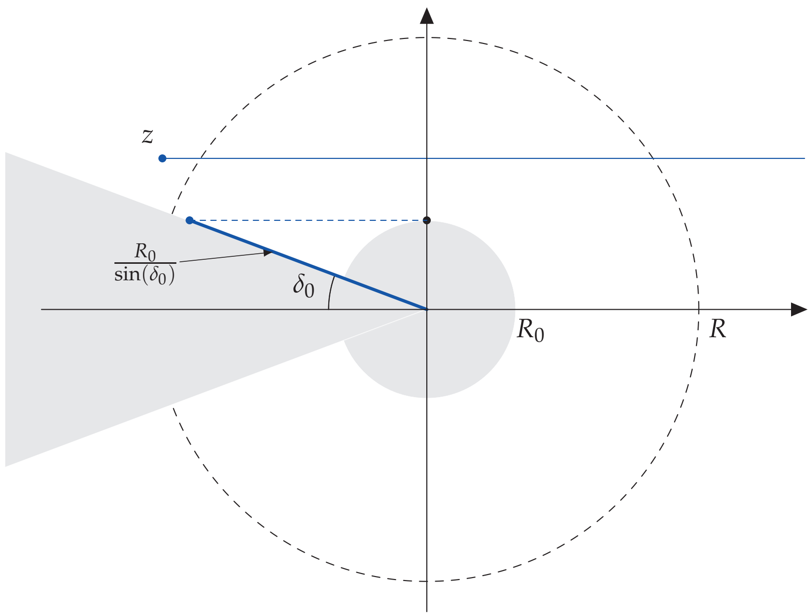

Figure 5.3: Geometric justification for the inequality $R \geq R_0/\sin(\delta_0)$.

We begin with the following general result in the theory of asymptotic integration.

Theorem 5.17. *Suppose that $F(z)$ satisfies Hypothesis $\mathbf{F}^+$, and let $w_{\sin}(z)$ be a non-trivial solution of the sine equation* (5.33)*. Then the singular Volterra integral equation*

$$w(z) = w_{\sin}(z) + \int_z^\infty \sin(t-z)F(t)w(t)dt, \tag{5.53}$$

where $z \in D^+(\delta, R)$ and the path of integration is $t - z = r$, $0 \le r < \infty$, has a unique solution $w(z)$ which is a solution of (5.51)*. Moreover, with $z = x + iy$ we have*

$$|w(z) - w_{\sin}(z)| \le M(y) \left\{ \exp\left[\int_x^\infty |F(s+iy)|\, ds \right] - 1 \right\}, \quad z \in D^+(\delta, R), \tag{5.54}$$

where $M(y) = \sup\limits_{s \ge x} |w_{\sin}(s+iy)|$.

Theorem 5.17 is used in [32, Chapter 7.4] without stating or proving it. However, the idea of the proof of Theorem 5.17 is based on Hille's work [31]. Hille, in fact, proves [32, Theorem 7.4.1], which is the reverse of Theorem 5.17. The latter has a direct role in the theory of asymptotic integration.

The following direct consequences of Theorem 5.17 show precisely the asymptotic correspondence between solutions of (5.51) and the three types of solutions e^{iz}, e^{-iz} and $\sin(z - z_0)$ of the equation (5.33).

Corollary 5.5. *Suppose that $F(z)$ satisfies Hypothesis $\mathbf{F}^+$. Then the perturbed sine equation* (5.51) *has unique linearly independent non-oscillatory solutions $E^+(z)$ and $E^-(z)$ asymptotic to e^{iz} and e^{-iz}, respectively, in $D^+(\delta, R)$ in the sense that*

$$E^+(z) = e^{iz}\left(1 + v_1(z)\right) \quad \text{and} \quad E^-(z) = e^{-iz}\left(1 + v_2(z)\right), \tag{5.55}$$

where

$$|v_s(z)| \le \exp\left[\int_z^\infty |F(t)|\, |dt| \right] - 1, \quad s = 1, 2, \tag{5.56}$$

and the path of integration is $t - z = r$, $0 \le r < \infty$, for each $z \in D^+(\delta, R)$.

From (5.52), the solutions $E^+(z)$ and $E^-(z)$ in Corollary 5.5 satisfy $E^+(z) = e^{iz}(1 + o(1))$ and $E^-(z) = e^{-iz}(1 + o(1))$ as $z \to \infty$ in $D^+(\delta, R)$. Corollary 5.5 is for the non-oscillatory solutions of (5.51). For the oscillatory solutions, we have the following result.

Corollary 5.6. *For any oscillatory solution $S(z)$ of* (5.51) *in $D^+(\delta, R)$, there exist two constants $b \neq 0$ and $z_0 = x_0 + iy_0$, such that*

$$S(z) = b\Big[\sin(z - z_0) + v(z)\Big], \tag{5.57}$$

where

$$|v(z)| \le \cosh(y - y_0) \left\{ \exp\left[\int_z^\infty |F(t)|\, |dt| \right] - 1 \right\}, \quad z \in D^+(\delta, R), \tag{5.58}$$

and the path of integration is $t - z = r$, $0 \le r < \infty$.

From (5.52), Corollary 5.6 shows that any oscillatory solution $S(z)$ of (5.51) is asymptotic to $\sin(z-z_0)$ in any horizontal strip in $D^+(\delta,R)$. Since the location and the number of zeros of $\sin(z-z_0)$ are known, it remains to estimate the location and the number of zeros of $S(z)$.

Before stating the next result, we define the following concepts: Let $\gamma>0$ be an arbitrary small constant, and let H_0 denote a half-plane

$$H_0=\{z:\mathrm{Re}(z)>\sigma_0\},$$

where $\sigma_0>0$ is chosen large enough so that both $H_0\subset D^+(\delta,R)$ and

$$\exp\left[\int_z^\infty |F(t)|\,|dt|\right]<1+\frac{\sin(\gamma)}{\cosh(\gamma)},\quad z\in H_0, \tag{5.59}$$

are satisfied. Observe that (5.59) follows from (5.52). In addition, we may assume that $z_0=x_0+iy_0$ satisfies $z_0-\gamma\in H_0$ and $z_0-\pi+\gamma\notin H_0$. For $k\geq 0$, let $Q_{k,\gamma}$ denote the square

$$Q_{k,\gamma}=\{z=x+iy:|x-x_0-k\pi|<\gamma,\ |y-y_0|<\gamma\}.$$

For any fixed $k\geq 0$, the point $z_0+k\pi$ is the center of the square $Q_{k,\gamma}$.

Lemma 5.6. *The function $S(z)$ in* (5.57) *is oscillatory in the half-plane H_0. Specifically, $S(z)$ has precisely one zero in each square $Q_{k,\gamma}$, and no other zeros in H_0. In addition, we have*

$$n\left(r,H_0,\frac{1}{S}\right)=\frac{r}{\pi}(1+o(1)),\quad r\to\infty, \tag{5.60}$$

where $n(r,H_0,1/S)$ counts only those zeros of $S(z)$ that lie in H_0 and $|z|\leq r$.

Remark 5.2. We can define Hypothesis $\mathbf{F}^-$ analogously to Hypothesis $\mathbf{F}^+$ by replacing $D^+(\delta,R)$ with the domain $D^-(\delta,R)$, which is the reflection of $D^+(\delta,R)$ with respect to the imaginary axis, and the path of integration is replaced with $t=z-r$, $0\leq r<\infty$. Then, all the results mentioned in Section 5.4.4 are true in the domain $D^-(\delta,R)$ under Hypothesis $\mathbf{F}^-$, provided that the path of integration in (5.56) and (5.58) is replaced with $t=z-r$, $0\leq r<\infty$, the half-plane H_0 and the squares $Q_{k,\gamma}$ are reflected with respect to the imaginary axis, see Section 3 in Paper IV.

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This survey contains new findings con complex linear differential equations with coefficients being either entire or analytic in the unit disc. We estimate the number of solutions that grow rapidly compared to the coefficients and satisfy certain asymptotic growth properties. In the second order case, new conditions on the coefficients are introduced to ensure that all solutions are of infinite order. This survey also contains new results on certain complex nonlinear differential equations solution, in which the asymptotic growth of solutions is obtained.

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