Complete coherence of random, nonstationary electromagnetic fields

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Despite a wide range of applications, coherence theory of random, nonstationary (pulsed or otherwise) electromagnetic fields is far from complete. In this work, we show that full coherence of a nonstationary vectorial field over a spatial volume and a spectral band is equivalent to the factorization of the cross-spectral density matrix in the spatiospectral variables. We further show that in this case the time-domain mutual coherence matrix factors in the spatiotemporal variables and the field is temporally fully coherent throughout the volume. The results of this work justify that certain expressions of random pulsed electromagnetic beams appearing in the literature can be called coherent-mode representations.

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Factorizability of coherence functions can be regarded as the very definition of full coherence [1]. In the context of classical, stationary, scalar optical fields, it has been shown long ago that complete spatial and temporal coherence in a volume implies that the mutual coherence function can be expressed as a product of two functions, one depending on the first space–time point only and the other on the second point [2, 3]. A similar factorization result was later derived in the space–frequency domain stating that full spatial coherence in a volume at a certain frequency is equivalent with the factorization of the cross-spectral density function in the spatial variables [3, 4]. Recently, these results were extended to stationary electromagnetic fields both in the frequency domain [5] and in the time domain [6], when full coherence is described in terms of the electromagnetic degree of coherence introduced in [7] (see also [8, 9]).

In this work, we analyze random, nonstationary, possibly pulsed, electromagnetic fields and prove that complete coherence over a spatial volume and spectral band (not at a single frequency) is equivalent with the factorization of the cross-spectral density matrix into a product of two vector functions which depend on separate spatiospectral points. We further show that when this condition is met, the field is also temporally fully coherent in the volume under consideration and the mutual coherence matrix factors in spatiotemporal variables. In free space the spectral and temporal vector functions obey the Helmholtz equation and the wave equation, respectively, and are divergence free. The results of this work also justify associating the term coherent-mode representation to some expansions of the coherence matrices that have appeared earlier in the literature [10].

The electric mutual coherence matrix, $\mathbf{\Gamma}(\mathbf{r}_1, \mathbf{r}_2, t_1, t_2)$, encompasses the second-order spatiotemporal coherence properties of a nonstationary (pulsed or nonpulsed) electromagnetic field at a pair of points, $\mathbf{r}_1$ and $\mathbf{r}_2$, and instants of time, t_1 and t_2. In general, the field may have three orthogonal field components and the elements of the coherence matrix are given by [10] (for stationary-field counterpart, see [3])

$$\Gamma_{jk}(\mathbf{r}_1, \mathbf{r}_2, t_1, t_2) = \langle E_j^*(\mathbf{r}_1, t_1) E_k(\mathbf{r}_2, t_2)\rangle, \quad (j,k) \in (x,y,z). \tag{1}$$

Above, $E_j(\mathbf{r}, t)$ and $E_k(\mathbf{r}, t)$ are two components of a vectorial complex analytic signal representing the electric field realization. In addition, the asterisk denotes complex conjugation and the angle brackets stand for ensemble averaging. In the case of a random pulse train, the ensemble may consist of different pulses or sequences of pulses to account for pulse jitter [11]. Henceforth we assume free space in which the propagation of the coherence-matrix elements is governed by the two wave equations

$$\nabla_p^2 \Gamma_{jk}(\mathbf{r}_1, \mathbf{r}_2, t_1, t_2) - \frac{1}{c^2}\frac{\partial^2}{\partial t_p^2}\Gamma_{jk}(\mathbf{r}_1, \mathbf{r}_2, t_1, t_2) = 0, \quad p \in (1,2), \tag{2}$$

where ∇_p operates on $\mathbf{r}_p$ and c is the speed of light in vacuum. Furthermore, it follows from the divergence condition of Maxwell's equations that each coherence matrix element satisfies the equation

$$\sum_j \partial_j^1 \Gamma_{jk}(\mathbf{r}_1, \mathbf{r}_2, t_1, t_2) = 0, \tag{3}$$

where ∂_j^1 operates on $\mathbf{r}_1$. A relation similar to Eq. (3) exists where the operator is replaced by ∂_k^2. The spectral spatial coherence properties at (angular) frequencies ω_1 and ω_2 are described by the cross-spectral density matrix, $\mathbf{W}(\mathbf{r}_1, \mathbf{r}_2, \omega_1, \omega_2)$, whose elements are written as

$$W_{jk}(\mathbf{r}_1, \mathbf{r}_2, \omega_1, \omega_2) = \langle E_j^*(\mathbf{r}_1, \omega_1) E_k(\mathbf{r}_2, \omega_2)\rangle, \tag{4}$$

where $E_j(\mathbf{r}, \omega)$ and $E_k(\mathbf{r}, \omega)$ are the Fourier transforms of the related (square integrable) time domain realizations. Owing to the aforementioned Fourier transform relationship, the space–frequency and space–time coherence matrices obey the integral relations [10]

$$\mathbf{W}(\mathbf{r}_1,\mathbf{r}_2,\omega_1,\omega_2) = \frac{1}{4\pi^2}\iint_{-\infty}^{\infty}\mathbf{\Gamma}(\mathbf{r}_1,\mathbf{r}_2,t_1,t_2) \times \exp[\mathrm{i}(-\omega_1 t_1+\omega_2 t_2)]\mathrm{d}t_1\mathrm{d}t_2, \tag{5}$$

$$\mathbf{\Gamma}(\mathbf{r}_1,\mathbf{r}_2,t_1,t_2) = \iint_{0}^{\infty}\mathbf{W}(\mathbf{r}_1,\mathbf{r}_2,\omega_1,\omega_2) \times \exp[-\mathrm{i}(-\omega_1 t_1+\omega_2 t_2)]\mathrm{d}\omega_1\mathrm{d}\omega_2. \tag{6}$$

Together with Eqs. (2) and (3), the integral relations imply the Helmholtz equations

$$\nabla_p^2 W_{jk}(\mathbf{r}_1,\mathbf{r}_2,\omega_1,\omega_2) + \left(\frac{\omega_p}{c}\right)^2 W_{jk}(\mathbf{r}_1,\mathbf{r}_2,\omega_1,\omega_2) = 0, \tag{7}$$

with $p \in (1,2)$, and the divergence condition

$$\sum_j \partial_j^1 W_{jk}(\mathbf{r}_1,\mathbf{r}_2,\omega_1,\omega_2) = 0. \tag{8}$$

Another condition is obtained by replacing ∂_j^1 with ∂_k^2. Equations (1) and (4) indicate that the matrices $\mathbf{\Gamma}(\mathbf{r}_1,\mathbf{r}_2,t_1,t_2)$ and $\mathbf{W}(\mathbf{r}_1,\mathbf{r}_2,\omega_1,\omega_2)$ are Hermitian in the sense that

$$\Gamma_{jk}(\mathbf{r}_1,\mathbf{r}_2,t_1,t_2) = \Gamma_{kj}^*(\mathbf{r}_2,\mathbf{r}_1,t_2,t_1), \tag{9}$$

$$W_{jk}(\mathbf{r}_1,\mathbf{r}_2,\omega_1,\omega_2) = W_{kj}^*(\mathbf{r}_2,\mathbf{r}_1,\omega_2,\omega_1). \tag{10}$$

The essential difference between the coherence properties of stationary and nonstationary fields is that for the former the different frequency components are uncorrelated while for the latter they may be partially or fully correlated. This important physical feature needs to be taken into account in deriving the conditions of complete coherence for nonstationary fields and sets the following analysis apart from that in [4–6].

We next consider the implications of the following inequality in the space–frequency domain

$$\langle |\sum_{p=1}^{N} a_p \Lambda_p(\mathbf{r}_p,\omega_p)|^2\rangle \geq 0, \tag{11}$$

which is valid for any N real or complex numbers a_p and complex random variables $\Lambda_p(\mathbf{r}_p,\omega_p)$, defined over any N position and frequency arguments $\mathbf{r}_p$ and ω_p within an observation domain D and frequency range Δ, respectively. We first choose $N = 2$, and

$$\Lambda_p(\mathbf{r}_p,\omega_p) = \delta_{p1}E_j(\mathbf{r}_p,\omega_p) + \delta_{p2}E_k(\mathbf{r}_p,\omega_p), \tag{12}$$

where the Kronecker deltas, δ_{p1} and δ_{p2}, assign to $\Lambda_p(\mathbf{r}_p,\omega_p)$ the j and k field components at $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively. By substituting Eq. (12) into inequality (11) and using the notation of Eq. (4), we obtain the relation

$$\begin{bmatrix} a_1^* & a_2^* \end{bmatrix} \begin{bmatrix} W_{jj}(\mathbf{r}_1,\mathbf{r}_1,\omega_1,\omega_1) & W_{jk}(\mathbf{r}_1,\mathbf{r}_2,\omega_1,\omega_2) \\ W_{kj}(\mathbf{r}_2,\mathbf{r}_1,\omega_2,\omega_1) & W_{kk}(\mathbf{r}_2,\mathbf{r}_2,\omega_2,\omega_2) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \geq 0. \tag{13}$$

The inequality (13) indicates that the related 2×2 matrix is Hermitian and nonnegative definite ([12], Sec. 13.5-3). Thus, the matrix has a nonnegative determinant ([12], Sec. 13.5-6) which, together with the Hermiticity property in Eq. (10), implies that

$$|W_{jk}(\mathbf{r}_1,\mathbf{r}_2,\omega_1,\omega_2)|^2 \leq W_{jj}(\mathbf{r}_1,\mathbf{r}_1,\omega_1,\omega_1)W_{kk}(\mathbf{r}_2,\mathbf{r}_2,\omega_2,\omega_2). \tag{14}$$

Therefore, we can normalize the spectral coherence matrix elements by defining

$$\mu_{jk}(\mathbf{r}_1,\mathbf{r}_2,\omega_1,\omega_2) = \frac{W_{jk}(\mathbf{r}_1,\mathbf{r}_2,\omega_1,\omega_2)}{[W_{jj}(\mathbf{r}_1,\mathbf{r}_1,\omega_1,\omega_1)W_{kk}(\mathbf{r}_2,\mathbf{r}_2,\omega_2,\omega_2)]^{1/2}}, \tag{15}$$

where

$$0 \leq |\mu_{jk}(\mathbf{r}_1,\mathbf{r}_2,\omega_1,\omega_2)| \leq 1, \tag{16}$$

for all $(j,k) \in (x,y,z)$. The lower (upper) limits correspond to complete noncorrelation (correlation) between the j and k field components of a nonstationary field at $(\mathbf{r}_1,\omega_1)$ and $(\mathbf{r}_2,\omega_2)$. Moreover, $\mu_{jk}(\mathbf{r}_1,\mathbf{r}_2,\omega_1,\omega_2)$ satisfies the relation

$$\mu_{jk}(\mathbf{r}_1,\mathbf{r}_2,\omega_1,\omega_2) = \mu_{kj}^*(\mathbf{r}_2,\mathbf{r}_1,\omega_2,\omega_1), \tag{17}$$

which immediately follows from Eqs. (10) and (15).

The degree of coherence for electromagnetic fields in the spectral domain has been introduced for stationary fields in [5] (see also [9]). For beam fields it physically describes the contrasts of spectral density and polarization modulations in interference [8]. Alternatively, it can be considered as a measure for the strengths of correlations between the orthogonal field components in two points at a single frequency [5]. Following the latter interpretation we can extend the quantity to the case of nonstationary three-component fields at two space–frequency points by defining

$$\mu^2(\mathbf{r}_1,\mathbf{r}_2,\omega_1,\omega_2) = \frac{\mathrm{tr}\,[\mathbf{W}(\mathbf{r}_1,\mathbf{r}_2,\omega_1,\omega_2)\mathbf{W}(\mathbf{r}_2,\mathbf{r}_1,\omega_2,\omega_1)]}{\mathrm{tr}\mathbf{W}(\mathbf{r}_1,\mathbf{r}_1,\omega_1,\omega_1)\,\mathrm{tr}\mathbf{W}(\mathbf{r}_2,\mathbf{r}_2,\omega_2,\omega_2)}, \tag{18}$$

where tr denotes the trace operation. The degree of coherence, $\mu\,(\mathbf{r}_1,\mathbf{r}_2,\omega_1,\omega_2)$, can be written in terms of the correlation coefficients in Eq. (15) as

$$\mu^2(\mathbf{r}_1,\mathbf{r}_2,\omega_1,\omega_2) = \frac{\sum_{jk}|\mu_{jk}(\mathbf{r}_1,\mathbf{r}_2,\omega_1,\omega_2)|^2 W_{jj}(\mathbf{r}_1,\mathbf{r}_1,\omega_1,\omega_1)W_{kk}(\mathbf{r}_2,\mathbf{r}_2,\omega_2,\omega_2)}{\sum_{jk}W_{jj}(\mathbf{r}_1,\mathbf{r}_1,\omega_1,\omega_1)W_{kk}(\mathbf{r}_2,\mathbf{r}_2,\omega_2,\omega_2)}. \tag{19}$$

In view of Eq. (19), the degree of coherence can be regarded as an intensity-weighted average of the (squared) magnitudes of the correlation coefficients paralleling the stationary field interpretation mentioned above. We further see that the field is incoherent at two space–frequency points, $\mu(\mathbf{r}_1,\mathbf{r}_2,\omega_1,\omega_2) = 0$, if and only if the field components are fully uncorrelated at these points, i.e., $\mu_{jk}(\mathbf{r}_1,\mathbf{r}_2,\omega_1,\omega_2) = 0$ for all $(j,k) \in (x,y,z)$. In addition, the field is considered fully coherent at two space–time points,

$$\mu(\mathbf{r}_1,\mathbf{r}_2,\omega_1,\omega_2) = 1, \tag{20}$$

if and only if the field components are fully correlated, i.e.,

$$|\mu_{jk}(\mathbf{r}_1,\mathbf{r}_2,\omega_1,\omega_2)| = 1, \tag{21}$$

for all possible (j,k) combinations.

We next consider the implications of complete coherence, stated by Eq. (20), on the functional form of the cross-spectral density matrix. To this end, we set $N = 3$ in inequality (11) and consider an additional orthogonal field component, E_l with

$l \in (x, y, z)$, at a third point $\mathbf{r}_3$ and frequency ω_3 in $\Lambda_p(\mathbf{r}_p, \omega_p)$. We thus write

$$\Lambda_p(\mathbf{r}_p, \omega_p) = \delta_{p1} E_j(\mathbf{r}_p, \omega_p) + \delta_{p2} E_k(\mathbf{r}_p, \omega_p) + \delta_{p3} E_l(\mathbf{r}_p, \omega_p). \quad (22)$$

Inserting Eq. (22) into Eq. (11), and utilizing Eqs. (4) and (15), we obtain

$$\begin{bmatrix} b_1^* & b_2^* & b_3^* \end{bmatrix} \begin{bmatrix} \mu_{jj}^{11} & \mu_{jk}^{12} & \mu_{jl}^{13} \\ \mu_{kj}^{21} & \mu_{kk}^{22} & \mu_{kl}^{23} \\ \mu_{lj}^{31} & \mu_{lk}^{32} & \mu_{ll}^{33} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \geq 0, \quad (23)$$

where b_p is an arbitrary complex number connected to a_p via $b_p = a_p(\delta_{p1} W_{jj}^{pp} + \delta_{p2} W_{kk}^{pp} + \delta_{p3} W_{ll}^{pp})$, $p \in (1, 2, 3)$. The abbreviations $\mu_{mn}^{pq} = \mu_{mn}(\mathbf{r}_p, \mathbf{r}_q, \omega_p, \omega_q)$ and $W_{mm}^{pp} = W_{mm}(\mathbf{r}_p, \mathbf{r}_p, \omega_p, \omega_p)$, with $(m, n) \in (j, k, l)$ and $(p, q) \in (1, 2, 3)$, were introduced for convenience. Inequality (23) implies that the related 3×3 matrix is Hermitian and nonnegative definite and, therefore, has a nonnegative determinant [12]

$$\begin{vmatrix} 1 & \mu_{jk}^{12} & \mu_{jl}^{13} \\ \mu_{kj}^{21} & 1 & \mu_{kl}^{23} \\ \mu_{lj}^{31} & \mu_{lk}^{32} & 1 \end{vmatrix} \geq 0, \quad (24)$$

where the diagonal elements equal unity since $\mu_{mm}^{pp} = 1$. By making use of Eq. (21) and the Hermiticity relation in Eq. (17), evaluation of the determinant results in

$$2\text{Re}[\mu_{jk}^{12} \mu_{kl}^{23} \mu_{lj}^{31}] \geq |\mu_{jk}^{12}|^2 + |\mu_{kl}^{23}|^2 + |\mu_{lj}^{31}|^2 - 1, \quad (25)$$

where Re denotes the real part.

We now assume that the electric field is completely coherent throughout a volume D and frequency range Δ, i.e., $\mu(\mathbf{r}_p, \mathbf{r}_q, \omega_p, \omega_q) = 1$ for all $(\mathbf{r}_p, \mathbf{r}_q) \in D$ and $(\omega_p, \omega_q) \in \Delta$. This implies that $|\mu_{jk}^{12}| = |\mu_{kl}^{23}| = |\mu_{lj}^{31}| = 1$ holds for any pair of space–frequency points $(1, 2, 3)$ and field components (j, k, l). Consequently, Eq. (25) reduces to

$$\text{Re}[\mu_{jk}^{12} \mu_{kl}^{23} \mu_{lj}^{31}] \geq 1. \quad (26)$$

In view of Eq. (21) the correlation coefficients can be written as

$$\mu_{mn}^{pq} = \exp(\mathrm{i}\varphi_{mn}^{pq}), \quad (27)$$

where the notation $\varphi_{mn}^{pq} = \varphi_{mn}(\mathbf{r}_p, \mathbf{r}_q, \omega_p, \omega_q)$ is employed for the phase factors which are real, by definition, and as a consequence of Eq. (17) satisfy

$$\varphi_{mn}^{pq} = -\varphi_{nm}^{qp}. \quad (28)$$

Upon substituting Eq. (27) in inequality (26), we obtain

$$\cos(\varphi_{jk}^{12} + \varphi_{kl}^{23} + \varphi_{lj}^{31}) \geq 1, \quad (29)$$

which is satisfied only if

$$\varphi_{jk}^{12} + \varphi_{kl}^{23} + \varphi_{lj}^{31} = 2\pi M, \quad (30)$$

where M is an integer. Recall from Eq. (22) that j, k, and l correspond to arbitrary field components at $(\mathbf{r}_1, \omega_1)$, $(\mathbf{r}_2, \omega_2)$, and $(\mathbf{r}_3, \omega_3)$, respectively. We now fix l as well as the related point $\mathbf{r}_3$ and frequency ω_3. In order to emphasize this we denote $l = L$, $\mathbf{r}_3 = \mathbf{r}_0$, and $\omega_3 = \omega_0$. It follows from Eqs. (27) and (30) that

$$\mu_{jk}(\mathbf{r}_1, \mathbf{r}_2, \omega_1, \omega_2) = \exp\{\mathrm{i}[-\varphi_{Lj}(\mathbf{r}_0, \mathbf{r}_1, \omega_0, \omega_1)]\} \times \exp\{\mathrm{i}[\varphi_{Lk}(\mathbf{r}_0, \mathbf{r}_2, \omega_0, \omega_2)]\}. \quad (31)$$

Inserting this result into Eq. (15) we find

$$W_{jk}(\mathbf{r}_1, \mathbf{r}_2, \omega_1, \omega_2) = \xi_j^*(\mathbf{r}_1, \omega_1)\, \xi_k(\mathbf{r}_2, \omega_2), \quad (32)$$

where

$$\xi_j(\mathbf{r}, \omega) = [W_{jj}(\mathbf{r}, \mathbf{r}, \omega, \omega)]^{1/2} \exp[\mathrm{i}\beta_j(\mathbf{r}, \omega)], \quad (33)$$

with $\beta_j(\mathbf{r}, \omega) = \varphi_{Lj}(\mathbf{r}_0, \mathbf{r}, \omega_0, \omega)$. The quantities r_0 and ω_0 as well as the index L are arbitrary parameters effectively setting the references in the three domains, thereby defining the phase function $\beta_j(\mathbf{r}, \omega)$.

It then follows that if a field is fully coherent over a spatial domain D and spectral band Δ, i.e., Eq. (20) holds for all pairs of spatiospectral points within these regions, the full cross-spectral density matrix with the elements described by Eqs. (32) and (33) can be written in the factored form

$$\mathbf{W}(\mathbf{r}_1, \mathbf{r}_2, \omega_1, \omega_2) = \boldsymbol{\mathcal{E}}^*(\mathbf{r}_1, \omega_1) \boldsymbol{\mathcal{E}}^{\mathrm{T}}(\mathbf{r}_2, \omega_2), \quad (34)$$

where T denotes transpose and

$$\boldsymbol{\mathcal{E}}(\mathbf{r}, \omega) = [\xi_x(\mathbf{r}, \omega), \xi_y(\mathbf{r}, \omega), \xi_z(\mathbf{r}, \omega)]^{\mathrm{T}}. \quad (35)$$

Conversely, inserting the factored matrix of Eq. (34) into Eq. (19), we recover the condition in Eq. (20). We therefore conclude that the factorization of the cross-spectral density matrix in a spatiospectral volume is equivalent with complete spatiospectral electromagnetic coherence. This is one of the main results of this work.

Equations (7) and (8) indicate that the complex vector function $\boldsymbol{\mathcal{E}}(\mathbf{r}, \omega)$ satisfies the Helmholtz equation

$$\nabla^2 \boldsymbol{\mathcal{E}}(\mathbf{r}, \omega) + \left(\frac{\omega}{c}\right)^2 \boldsymbol{\mathcal{E}}(\mathbf{r}, \omega) = 0, \quad (36)$$

and the divergence condition

$$\nabla \cdot \boldsymbol{\mathcal{E}}(\mathbf{r}, \omega) = 0. \quad (37)$$

These two results imply that a random, nonstationary, electric field or pulse train does not necessitate, in the limit of complete coherence, the use of the cross-spectral density matrix but can be treated in terms of the field-level vector function $\boldsymbol{\mathcal{E}}(\mathbf{r}, \omega)$. The ensemble in Eq. (4) may be viewed as consisting of identical realizations, each equal to $\boldsymbol{\mathcal{E}}(\mathbf{r}, \omega)$.

Consider next some consequences of the above results. We note that in the literature the cross-spectral density matrix of a general partially polarized and partially coherent nonstationary electromagnetic field has been described as a coherent-mode decomposition of the form [10]

$$\mathbf{W}(\mathbf{r}_1, \mathbf{r}_2, \omega_1, \omega_2) = \sum_n \lambda_n \boldsymbol{\Phi}_n^*(\mathbf{r}_1, \omega_1) \boldsymbol{\Phi}_n^{\mathrm{T}}(\mathbf{r}_2, \omega_2), \quad (38)$$

where λ_n are nonnegative real coefficients and $\boldsymbol{\Phi}_n(\mathbf{r}, \omega)$ are orthonormal mode functions obeying a homogeneous Fredholm equation of the second kind. In view of the result in Eq. (34), the terms in Eq. (38) represent fully coherent fields which are mutually uncorrelated. This observation justifies the term coherent-mode representation for the sum in Eq. (38).

We now turn our attention to the form of the time-domain coherence matrix, $\mathbf{\Gamma}(\mathbf{r}_1, \mathbf{r}_2, t_1, t_2)$, under the condition of full spatiospectral coherence stated by Eq. (20), which is equivalent with the factorization of the cross-spectral density matrix. Substituting Eq. (34) in Eq. (6) results in

$$\mathbf{\Gamma}(\mathbf{r}_1, \mathbf{r}_2, t_1, t_2) = \tilde{\boldsymbol{\mathcal{E}}}^*(\mathbf{r}_1, t_1)\tilde{\boldsymbol{\mathcal{E}}}^{\mathrm{T}}(\mathbf{r}_2, t_2), \tag{39}$$

where

$$\tilde{\boldsymbol{\mathcal{E}}}(\mathbf{r}, t) = \int_0^\infty \boldsymbol{\mathcal{E}}(\mathbf{r}, \omega) \exp(-\mathrm{i}\omega t)\mathrm{d}\omega. \tag{40}$$

Therefore, the electric mutual coherence matrix factors into two parts depending exclusively on the points $(\mathbf{r}_1, t_1)$ and $(\mathbf{r}_2, t_2)$, respectively. The time-domain version of the spectral degree of coherence in Eq. (18) can be introduced as

$$\gamma^2(\mathbf{r}_1, \mathbf{r}_2, t_1, t_2) = \frac{\mathrm{tr}\,[\mathbf{\Gamma}(\mathbf{r}_1, \mathbf{r}_2, t_1, t_2)\mathbf{\Gamma}(\mathbf{r}_2, \mathbf{r}_1, t_2, t_1)]}{\mathrm{tr}\mathbf{\Gamma}(\mathbf{r}_1, \mathbf{r}_1, t_1, t_1)\,\mathrm{tr}\mathbf{\Gamma}(\mathbf{r}_2, \mathbf{r}_2, t_2, t_2)}, \tag{41}$$

and we readily find that $\gamma(\mathbf{r}_1, \mathbf{r}_2, t_1, t_2) = 1$ holds for the field represented by the factored matrix of Eq. (39). It follows that a spatially (within D) and spectrally (within Δ) completely coherent nonstationary field is spatiotemporally fully coherent at all instants of time. This is in contrast to stationary scalar or electromagnetic fields where full spatial coherence at all frequencies does not imply complete temporal coherence [13, 14].

We also observe that due to Eqs. (36), (37), and (40), $\tilde{\boldsymbol{\mathcal{E}}}(\mathbf{r}, t)$ obeys the wave equation and is divergence-free in free space. These are respectively written as

$$\nabla^2\tilde{\boldsymbol{\mathcal{E}}}(\mathbf{r}, t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\tilde{\boldsymbol{\mathcal{E}}}(\mathbf{r}, t) = 0, \tag{42}$$

$$\nabla \cdot \tilde{\boldsymbol{\mathcal{E}}}(\mathbf{r}, t) = 0. \tag{43}$$

Thus, in time domain, a (random) fully coherent nonstationary field can be treated by considering the vector function $\tilde{\boldsymbol{\mathcal{E}}}(\mathbf{r}, t)$ and the analysis of the full coherence matrix is not necessary.

Using Eq. (38) in Eq. (6) results in the following decomposition of the time-domain coherence matrix of a nonstationary electromagnetic field (see also [10])

$$\mathbf{\Gamma}(\mathbf{r}_1, \mathbf{r}_2, t_1, t_2) = \sum_n \lambda_n' \boldsymbol{\psi}_n^*(\mathbf{r}_1, t_1)\boldsymbol{\psi}_n^{\mathrm{T}}(\mathbf{r}_2, t_2), \tag{44}$$

where $\lambda_n' = 2\pi\lambda_n$ are the weights of the orthonormal time domain mode functions

$$\boldsymbol{\psi}_n(\mathbf{r}, t) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \boldsymbol{\psi}_n(\mathbf{r}, \omega) \exp(-\mathrm{i}\omega t)\mathrm{d}\omega. \tag{45}$$

Equation (44) expresses the time-domain coherence function as a sum of coherence functions representing mutually uncorrelated, spatiotemporally fully coherent fields. Consequently, we may regard it as a coherent-mode representation of the time-domain mutual coherence matrix.

Finally, we note that the case of scalar fields is encountered as a special case of the above analysis if only a single field component is retained. Therefore, complete spatiospectral and spatiotemporal coherence of a nonstationary scalar field over a volume is equivalent with the factorizations of the spectral and temporal coherence functions.

In summary, our analysis shows that a random nonstationary electromagnetic field is completely coherent in a spatiospectral volume in the sense of Eq. (20), if and only if, the cross-spectral density matrix factors in the space–frequency variables. This also implies full temporal coherence and the factorization of the mutual coherence matrix. In free space the vectorial factor functions obey the propagation equations of the corresponding domains and are divergence-free. The results also justify the extension of the term coherent mode to cover nonstationary electromagentic fields.

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