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TONI VESIKKO

ON ANALYTIC AND GEOMETRIC PROPERTIES OF CONFORMAL MAPS OF THE UNIT DISC IN CERTAIN FUNCTION SPACES

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ACADEMIC DISSERTATION

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Toni Vesikko
On analytic and geometric properties of conformal maps of the unit disc in certain function spaces
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ABSTRACT

This thesis contains various new results for univalent functions of the unit disc of the complex plane, partly via considering an extension to a known univalence criterion by Becker, and partly via studying the univalent functions of certain weighted spaces of analytic functions. This leads to both geometric and analytic considerations on the properties of univalent functions.

The case of Nehari's univalence criterion with a linear error, originally posed by Chuaqui and Stowe, is considered in the context of Becker's famous univalence criterion. We find that Becker's criterion with a linear error guarantees the univalence of locally univalent functions in certain horodiscs of the unit disc. Conversely, it's shown that univalence in certain horodiscs guarantees a type of Becker's criterion with a larger upper bound. We also consider generalizations of these results to locally univalent harmonic functions.

The univalent functions of certain spaces of analytic functions are studied to attain new estimates and inclusions between the spaces. In the case of the Hardy space H^p, we improve a known characterization and present its extension to the radially weighted Bergman space A^p_ω. Norm inequalities depending on the inducing radial weight are presented for the weighted Bergman space. We establish a variety of inequalities for the conformal maps of certain weighted Dirichlet and Besov type spaces as well as the Hardy-Littlewood space HL_p. In addition, the relation of these inequalities to certain weighted integral involving the maximum modulus and some geometrically defined function spaces is considered.

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Joensuu, August 9, 2024

Toni Vesikko

LIST OF PUBLICATIONS

This thesis consists of the present review of the author's work in the field of mathematical analysis comprised of the following selection of the author's publications:

I J.-M. Huusko and T. Vesikko, "On Becker's univalence criterion," *J. Math. Anal. Appl.* **458**, 781–794 (2018).

II F. Pérez-González, J. Rättyä, and T. Vesikko, "Integral means of derivatives of univalent functions in Hardy spaces," *Proc. Amer. Math. Soc.* **151**, 611–621 (2023).

III F. Pérez-González, J. Rättyä, and T. Vesikko, "Norm inequalities for weighted Dirichlet spaces with applications to conformal maps," submitted https://arxiv.org/abs/2201.06122

Throughout the overview, these papers will be referred to by Roman numerals.

AUTHOR'S CONTRIBUTION

The publications selected in this dissertation are original research papers on mathematical analysis. All authors have made an approximately equal contribution.

TABLE OF CONTENTS

1 Introduction

The study of univalent functions, also known as conformal mappings, or simply maps, is an important and classical subject in the field of mathematical analysis. This research has over the last century focused on studying some of the profound qualities of normalized univalent (or Schlicht) functions, but has also kept broadening in various ways to now contain many new branches and areas of interest.

Time has shown that the concepts present within the study of univalent functions of the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ often have some kind of a connection to differential equations. One of the most well-known examples is the linear differential equation

$$f'' + Af = 0, \tag{1.1}$$

with an analytic coefficient A, for which we have the relation $2A = S_f$, where S_f is the Schwarzian derivative of $f = g/h$ and $g, h \in \mathcal{H}(\mathbb{D})$ are two linearly independent non-trivial solutions of (1.1). The famous univalence criterion by Nehari [43] states that if

$$|S_f(z)|(1 - |z|^2)^2 \leq 2, \quad z \in \mathbb{D}, \tag{1.2}$$

for a locally univalent meromorphic function, then f is univalent in $\mathbb{D}$, or $f \in \mathcal{U}$. Due to the aforementioned relation between the analytic coefficient A and the Schwarzian of f, this criterion has also served as a natural path to studying the zero separation and oscillation of solutions of linear differential equations.

In relation to Nehari's criterion, another famous univalence criterion by Becker [5] states that if an analytic function f with $f'(0) \neq 0$ satisfies

$$|zP_f(z)|(1 - |z|^2) \leq 1, \quad z \in \mathbb{D}, \tag{1.3}$$

then f is univalent in $\mathbb{D}$. Chuaqui and Stowe showed in [10] that replacing the right-hand side of (1.2) with any continuous function decaying to one (with the substitution $2A = S_f$) slower than at a linear rate allows oscillation for some non-trivial solution of (1.1). Naturally this raised the question whether or not (1.2) with an additive linear error guarantees any kind of univalence for f, a question which was partially answered in [24]. In the same sense one might ask whether (1.3) with a linear error guarantees any kind of univalence or other neat properties for f. This has been the primary focus in Paper **I**, in which partial answers are given. In addition, analogues of the main results of the paper are considered for harmonic functions.

As with most types and classes of functions, inclusions to well-known function spaces and estimates regarding their norms are important topics of interest. This is especially true for conformal maps, whose behaviour in most classical function spaces has been studied to some extent. In Paper **II**, a known characterization of univalent functions of the Hardy space H^p in terms of the integral

$$I_{p,q}(f) = \int_0^1 M_q^p(r, f')(1-r)^{p\left(1-\frac{1}{q}\right)}\, dr + |f(0)|^p,$$

where $0 < p \leq q < \infty$, is studied. Its counterpart in the setting of the Bergman space A^p_ω induced by a radial weight, that is, an integrable function $\omega : \mathbb{D} \to [0, \infty)$ such that $\omega(z) = \omega(|z|)$ for $z \in \mathbb{D}$, is also discussed. In addition, the characterization for the Hardy space H^p is considered in the context of the close-to-convex functions, an important subclass of $\mathcal{S}$.

In Paper **III**, various norm inequalities for analytic and conformal mappings are studied in the context of weighted Bergman and Dirichlet type function spaces. While some of the results are true for all analytic functions, others serve as characterizations of conformal mappings of the studied weighted function spaces and classes via asymptotic equalities. Additionally, the relation of the integral quantity

$$J^p_\omega(f) = \int_0^1 M^p_\infty(r, f)\omega(r)\, dr,$$

involving the maximum modulus function $M^p_\infty(r, f)$ of f to the norms of the aforementioned spaces is considered. Some results are compared with the geometrically defined function spaces H^p_ω and S^p_ω defined as the sets of analytic functions for which

$$\|f\|^p_{H^p_\omega} = \int_0^1 \left(\int_{D(0,r)} \Delta|f|^p \, dA \right) \omega(r)\, dr = \int_{\mathbb{D}} \Delta|f|^p \widehat{\omega}\, dA,$$

and

$$\|f\|^p_{S^p_\omega} = \int_0^1 \left(\int_{D(0,r)} |f'|^2 \, dA \right)^{\frac{p}{2}} \omega(r)\, dr = \int_0^1 \mathrm{Area}(f(D(0,r)))^{\frac{p}{2}} \omega(r)\, dr,$$

are respectively finite. These definitions originate from [31, 42] for the classical weight $\omega_\alpha(z) \asymp (1 - |z|^2)^\alpha$. As all of the studied function spaces and integral quantities are radially weighted, proving most of the results requires analysis specifically on how much needs to be assumed from the inducing weight function in order to attain certain key estimates.

The remainder of this overview is organized as follows. In Chapter 2, the classical analysis and research history of univalent functions are discussed and the necessary concepts for placing the upcoming research results into their proper context are introduced. Chapter 3 consists of discussion on relevant function spaces and analysis of the weight functions which induce the weighted versions of the said spaces. We also briefly discuss existing research regarding the univalent functions of these functions spaces. To conclude, Chapter 4 is comprised of the summaries of Papers **I-III** in separate sections.

2 Univalent functions

In this chapter we will cover the basic theory of univalent functions necessary for the purposes of this thesis, as well as delve into some related topics to place the research conducted in this thesis in its proper context.

We begin by defining some basic notation that will be used for the entirety of this thesis. The complex plane is denoted by $\mathbb{C}$. We denote its unit disc by $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and the unit circle by $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. We denote the Euclidean disc of center $a \in \mathbb{C}$ and radius $r > 0$ by $D(a,r) = \{z \in \mathbb{C} : |z-a| < r\}$. The conformal automorphisms of the unit disc $\mathbb{D}$ are defined by $\varphi_a(z) = \lambda \frac{z-a}{1-\bar{a}z}$ for all $z, a \in \mathbb{D}$ and a unimodular constant $\lambda \in \mathbb{T}$. The pseudohyperbolic distance of two points $a, z \in \mathbb{D}$ is $\rho(a,z) = |\varphi_a(z)|$ and the pseudohyperbolic disc of center $a \in \mathbb{D}$ and radius $\delta > 0$ is denoted by $\Delta(a,\delta) = \{z \in \mathbb{D} : \rho(a,z) < \delta\}$. The hyperbolic metric in $\mathbb{D}$ is defined as

$$d_H(z,w) = \inf_{\gamma} \int_{\gamma} \frac{|dz|}{1-|z|^2} = \frac{1}{2} \log \frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|}, \quad z, w \in \mathbb{D},$$

where the infimum is taken over all parametrized smooth curves γ joining z and w.

2.1 CLASSICAL THEORY

For the fundamental theory of univalent functions, the classical monographs [14] and [56] are excellent sources and are also utilized in this section. Let D be a domain of the complex plane $\mathbb{C}$. An injective analytic function $f : D \to \mathbb{C}$ is called a univalent function. We denote the class of all univalent functions of the unit disc by $\mathcal{U}$. The class of Schlicht functions $\mathcal{S}$ is comprised of functions $f \in \mathcal{U}$ normalized such that $f(0) = 0$ and $f'(0) = 1$. If $f \in \mathcal{U}$, then $(f - f(0))/f'(0)$ belongs to $\mathcal{S}$. Clearly functions in $\mathcal{S}$ are of the form $f(z) = z + \widehat{f}(2)z^2 + \cdots = z + \sum_{k=2}^{\infty} \widehat{f}(k)z^k$, $z \in \mathbb{D}$, where we use the notation $a_n = \widehat{f}(n)$ for the Maclaurin coefficients.

If a function $f : \mathbb{D} \to \mathbb{C}$ is analytic at $z_0 \in \mathbb{D}$ such that $f'(z_0) \neq 0$, then f is univalent in some neighborhood of z_0 [62, p.198]. Therefore, a function f analytic in $\mathbb{D}$ is locally univalent if its Jacobian $J_f = |f'|^2$ is non-vanishing. A univalent function is also known as a conformal mapping, the geometric rationale for which can be found in [14, pp. 6]. We write $f \in U_{\text{loc}}^{A}$ for the class of analytic functions locally univalent in $\mathbb{D}$. A function f is meromorphic in $\mathbb{D}$ if at each point of the unit disc f is either analytic or has a pole. Furthermore, a function f meromorphic in $\mathbb{D}$ is locally univalent, denoted by $f \in U_{\text{loc}}^{M}$, if and only if its spherical derivative $f^{\#}(z) = |f'(z)|/(1+|f(z)|^2)$ is non-vanishing.

The study of univalent functions of the unit disc, and especially the normalized Schlicht functions, has been an active and important topic in the mathematical research of function theory. It is well known that the Koebe function

$$k(z) = \frac{z}{(1-z)^2} = \sum_{k=1}^{\infty} nz^n = \frac{1}{4}\left(\frac{1+z}{1-z}\right)^2 - \frac{1}{4}, \quad z \in \mathbb{D},$$

maps the unit disc $\mathbb{D}$ conformally onto the slit domain $\mathbb{C} \setminus (-\infty, \frac{1}{4}]$. One way to see this is the fact that the function $z \mapsto (1+z)/(1-z)$ maps the unit disc conformally onto the right half-plane $\{z \in \mathbb{C} : \Re z > 0\}$. The Koebe function is, in many sense, the maximal function of the class $\mathcal{S}$. One of the most fundamental estimates for $f \in \mathcal{S}$ shows that

$$\left| \frac{zf''(z)}{f'(z)} - \frac{2|z|^2}{1-|z|^2} \right| \leq \frac{4|z|}{1-|z|^2}, \quad z \in \mathbb{D}, \tag{2.1}$$

see [56, Lemma 1.3]. From this, the important Koebe growth and distortion theorem is deduced.

Theorem 2.1. *[56, Theorem 1.6] Let $f \in \mathcal{S}$. Then the following estimates hold for $z \in \mathbb{D}$.*

(i)

$$k'(-|z|) = \frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3} = k'(|z|),$$

(ii)

$$-k(-|z|) = \frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2} = k(|z|),$$

(iii)

$$|z|\frac{-k'(-|z|)}{k(-|z|)} = \frac{1-|z|}{1+|z|} \leq \left| z\frac{f'(z)}{f(z)} \right| \leq \frac{1+|z|}{1-|z|} = |z|\frac{k'(|z|)}{k(|z|)}.$$

In each case equality holds if and only if f is a suitable rotation of the Koebe function.

The Koebe distortion theorem already provides insight to the role of Koebe function as the maximal function of the class $\mathcal{S}$. It is also used in deriving the Koebe-1/4-Theorem [56, pp. 22], which states that $D(0, 1/4) \subset f(\mathbb{D})$ for all $f \in \mathcal{S}$. The study of univalent functions in general contains multiple geometric topics of interest. The distance of a point $f(z) = w$ from the boundary of the image domain $\partial f(\mathbb{D})$ is $\operatorname{dist}(w, \partial f(\mathbb{D})) = \inf_{\xi \in \partial f(\mathbb{D})} |w - \xi|$. It can be deduced from the Koebe distortion theorem [56, Corollary 1.4] that

$$\frac{1}{4}|f'(z)|(1-|z|^2) \leq \operatorname{dist}(w, \partial f(\mathbb{D})) \leq |f'(z)|(1-|z|^2), \tag{2.2}$$

for $f \in \mathcal{U}$, and in particular $\frac{1}{4} \leq \operatorname{dist}(0, \partial f(\mathbb{D})) \leq 1$ for $f \in \mathcal{S}$, further underlining the Koebe-1/4-theorem for Schlicht functions. It turns out that the geometric and analytic properties of conformal maps are often linked with one another, which we will briefly discuss later.

The Maclaurin coefficients of the functions $f \in \mathcal{S}$ have also been heavily studied. The famous Bieberbach conjecture states that, for a Schlicht function $f \in \mathcal{S}$ and $n \in \mathbb{N}$, the Maclaurin coefficients satisfy $|\widehat{f}(n)| \leq n$ with strict inequality unless f is a rotation of the Koebe function. Note that this means that the Koebe function has the maximal coefficients in the class $\mathcal{S}$. This conjecture was first presented by Bieberbach in 1916 in a footnote of paper [8] in which he proved the inequality for the second coefficient $\widehat{f}(2)$. Löwner proceeded to prove it for the third coefficient $\widehat{f}(3)$ in 1923 [41] introducing the Löwner differential equation and Löwner chains. The initial good estimate for all coefficients was soon after provided by Littlewood in 1925 [39], showing that $|\widehat{f}(n)| < en$, $n \in \mathbb{N}$. These findings were followed by decades of research, during which the conjecture was eventually proved for the fourth [19], fifth [45] and sixth [44] coefficients. In addition, the general estimate by Littlewood was improved multiple times, see for example [56, pp. 25] and the references therein. Eventually in 1985, Louis de Branges published a proof of the Bieberbach conjecture [11], leading to the result also being known as de Branges's theorem. It should also be noted that Fitzgerald and Pommerenke completed and published their proof of the conjecture later during the same year [16].

The study of univalent functions still contains various open questions, the most famous of which is widely considered to be the Brennan conjecture. Consider a simply connected domain $\Omega \subset \mathbb{C}$ with at least two boundary points in the extended complex plane. Let $\varphi : \Omega \to \mathbb{D}$ a conformal mapping onto $\mathbb{D}$ with $\varphi(w) = z \in \mathbb{D}$ and let $p \in \mathbb{R}$. Then the conjecture states that the integral

$$\int_{\Omega} |\varphi'(w)|^p \, dA(w)$$

converges for $4/3 < p < 4$. The integral was already known to converge for $4/3 < p < 3$ when Brennan conjectured the problem in 1978 [9]. Some improvement has since been made raising the upper bound of p, but the conjecture remains open.

2.2 DIFFERENTIAL EQUATIONS AND UNIVALENCE CRITERIA

The study of univalent functions and univalence criteria has shown to be in part closely related to the study of differential equations. It is known that for the differential equation

$$f'' + Af = 0, \tag{2.3}$$

where A is analytic, and any two of its linearly independent solutions g and h such that $f = g/h$, we have $2A = S_f$ analytic in $\mathbb{D}$, where S_f is the Schwarzian derivative of f. For $f \in U^A_{\text{loc}}$ the quantity $P_f = (\log f')' = f''/f'$ is called the pre-Schwarzian derivative of f. The pre-Schwarzian can also be derived from the Jacobian $J_f = |f'|^2$ with $P_f = \frac{\partial}{\partial z}(\log J_f)$, which perhaps emphasizes its geometric nature better. For $f \in U^M_{\text{loc}}$ the Schwarzian derivative is related to its pre-Schwarzian derivative by the relation

$$S_f = P_f' - \frac{1}{2}P_f^2 = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2.$$

The Schwarzian derivative is identically zero if and only if f is a Möbius transformation, thus it can vaguely be thought of as a measure of how much f differs from being a Möbius transformation.

The famous Nehari univalence criterion [43, Theorem 1] establishes a sufficient condition for the univalence of locally univalent meromorphic functions in terms of the Schwarzian derivative. Namely, if $f \in U^M_{\text{loc}}$ satisfies

$$|S_f(z)|(1 - |z|^2)^2 \leq \delta, \quad z \in \mathbb{D}, \tag{2.4}$$

for $\delta = 2$, then f is univalent in $\mathbb{D}$. This criterion is often seen with the earlier mentioned substitution $2A = S_f$ and $\delta = 1$ due to the Schwarzian derivative being intimately related to the solutions of (2.3). A converse of this theorem by Kraus [36] states that (2.4) with $\delta = 6$ is a necessary condition for the univalence of f. Nehari's univalence criterion is sharp by an example by Hille [30], which shows that for each

$\delta > 2$ there exists a function f analytic in the unit disc such that f satisfies (2.4) and attains the value 1 infinitely many times. Explicitly, Hille considered the function

$$f(z) = \left(\frac{1-z}{1+z}\right)^{\gamma i}, \quad z \in \mathbb{D},$$

where γ is a real constant, the branch of the complex power function is fixed with $f(0) = 1$, and the right-hand constant in (2.4) is $\delta = 2(1+\gamma^2)$. An example by Schwarz [59, p.162] also shows that, for each $\gamma > 0$, the functions

$$f(z) = \sqrt{1-z^2}\,\sin\left(\gamma \log \frac{1+z}{1-z}\right), \quad A(z) = \frac{1+4\gamma^2}{(1-z^2)^2}, \quad z \in \mathbb{D},$$

satisfy (2.3) and (2.4), latter with $\delta = 1+4\gamma^2$, but f has infinitely many zeroes in $\mathbb{D}$. Furthermore, by careful inspection of the proof of [59, Theorem 1] by Schwarz, it can be seen that for any non-trivial solution of (2.3) with $f(z_1) = f(z_2)$ and $z_1 \neq z_2$ we have

$$\max_{\xi \in \langle z_1, z_2 \rangle} |S_f(\xi)|(1-|\xi|^2)^2 > 2, \quad z_1, z_2 \in \mathbb{D}, \tag{2.5}$$

where the maximum is taken over the hyperbolic segment $\langle z_1, z_2 \rangle = \{\varphi_{z_1}(\varphi_{z_1}(z_2)t) : 0 \leq t \leq 1\}$ joining z_1 and z_2. Note that (2.5) implies the following: If (2.4) holds in an annulus $r_0 < |z| < 1$ for $\delta = 2$ and for some $0 < r_0 < 1$, then f has finite valence in $\mathbb{D}$, see [59, Corollary 1].

In relation to Nehari's univalence criterion, Chuaqui and Stowe showed in [10] that replacing the right-hand side constant in (2.4) with any continuous function $\beta : [0,1) \to (0,\infty)$ decaying to one slower than at a linear rate allows oscillatory behaviour for the solutions of (2.3).

Theorem 2.2. *[10, Theorem 5] If $\beta : [0,1) \to (0,\infty)$ is continuous and $\lim_{r\to 1} \frac{\beta(r)-1}{1-r} = \infty$, then there is a holomorphic function A in $\mathbb{D}$ satisfying $|A(z)| \leq \beta(|z|)/(1-|z|^2)^2$ for all $z \in \mathbb{D}$ such that some nontrivial solution of $f'' + Af = 0$ has infinitely many zeroes.*

Following this, Chuaqui and Stowe also posed the natural question of whether or not Nehari's condition with a linear error, that is,

$$|A(z)|(1-|z|^2)^2 \leq 1 + C(1-|z|), \quad z \in \mathbb{D}, \tag{2.6}$$

where $C > 0$, would imply finite oscillation of the solutions of (2.3). Some recent progress regarding this question has been made in [24]. Steinmetz [61, p. 328] showed earlier that if (2.6) holds, then f is a normal function, that is, the family $\{f \circ \varphi_a : a \in \mathbb{D}\}$, where φ_a is the unit disc automorphism, is normal in the sense of Montel [58, Chapters 2 and 3]. Equivalently, $\|f\|_{\mathcal{N}} = \sup_{z\in\mathbb{D}} f^{\#}(z)(1-|z|^2) < \infty$, see [37], and we write $f \in \mathcal{N}$.

Now let g analytic in $\mathbb{D}$. Then, by the Cauchy integral formula

$$|g'(z)|(1-|z|^2)^2 \leq 4 \max_{|\zeta| = \frac{1+|z|^2}{2}} |g(\zeta)|(1-|\zeta|^2), \quad z \in \mathbb{D}.$$

Consequently, the inequality

$$\|S_f\|_{H_2^\infty} \leq 4\|P_f\|_{H_1^\infty} + \frac{1}{2}\|P_f\|_{H_1^\infty}^2$$

holds. Here, for $0 < p < \infty$, we denote $\|g\|_{H_p^\infty} = \sup_{z\in\mathbb{D}} |g(z)|(1-|z|^2)^p$. Thus, both the conditions (2.4) and (2.6) hold if $|P_f(z)|(1-|z|^2)$ is sufficiently small for $z \in \mathbb{D}$. Note also that by [55, p. 133], we conversely have

$$\|P_f\|_{H_1^\infty} \leq 2 + 2\sqrt{1 + \frac{1}{2}\|S_f\|_{H_2^\infty}}.$$

In a similar fashion to Nehari's criterion, the well-known univalence criterion by Becker [5, Korollar 4.1] states that if an analytic function of the unit disc with $f'(0) \neq 0$ satisfies

$$|zP_f(z)|(1-|z|^2) \leq \rho, \quad z \in \mathbb{D}, \tag{2.7}$$

for $\rho = 1$, then f is univalent in $\mathbb{D}$. For $\rho > 1$, condition (2.7) does not guarantee the univalence of f [6, Satz 6]. Note that the inequality holds for $\rho = 6$ by (2.1). This is the converse of Becker's criterion providing another necessary condition for the univalence of f. Becker's univalence criterion also has a deep connection to the theory of differential equations, as the tools and methods used by Becker to initially prove the theorem originate from Löwner's differential equation and his 1923 paper [41].

If the assumption on the derivative of f is omitted in Becker's criterion and we assume only the analyticity of f, the left side of (2.7) reduces to $|P_f(z)|(1-|z|^2)$ [6]. Thus, in relation to the Chuaqui-Stowe question, it is natural to consider whether Becker's criterion with a linear error, that is,

$$|P_f(z)|(1-|z|^2) \leq 1 + C(1-|z|), \quad z \in \mathbb{D}, \tag{2.8}$$

for $C > 0$ would also guarantee any kind of univalence or bounded valence for the functions it holds for. Indeed, this has been the primary focus in Paper **I**.

2.3 IMPORTANT SUBCLASSES

There are some subclasses of univalent functions which play a key role in the geometric study of univalent functions of the unit disc. A domain $D \subset \mathbb{C}$ is starlike with respect to a point $z_0 \in D$, if for each $z \in D$ the line segment $[z, z_0]$ is contained in D. A domain is convex if it is starlike with respect to each of its points. A conformal mapping $f : \mathbb{D} \to D$ is starlike if the set D is starlike with respect to the origin, and similarly f is convex if D is a convex set. The subclass of $\mathcal{S}$ consisting of convex functions is denoted by $\mathcal{C}$ and the subclass consisting of starlike functions is denoted by $\mathcal{S}^*$. Clearly we have the inclusions $\mathcal{C} \subset \mathcal{S}^* \subset \mathcal{S}$.

A closely related class of analytic functions g with a positive real part such that $g(0) = 1$ is denoted by $\mathcal{P}$. This class of analytic functions has proven useful in noteworthy characterizations of both the starlike and convex subclasses of $\mathcal{S}$. For instance, by [14, Theorem 2.10], a function analytic in $\mathbb{D}$ normalized such that $f(0) = 0$ and $f'(0) = 1$ belongs to $\mathcal{S}^*$ if and only if $zf'(z)/f(z) \in \mathcal{P}$. Similarly by [14, Theorem 2.11], an analytic function normalized in the same way as above belongs to $\mathcal{C}$ if and only if $(1 + zf''(z)/f'(z)) \in \mathcal{P}$.

Another important geometric subclass of interest is the class of close-to-convex functions introduced by Kaplan [34]. A function f analytic in $\mathbb{D}$ is close-to-convex if there exists a convex function g such that the real part of f'/g' is strictly positive in $\mathbb{D}$. Although this definition may seem arbitrary at first, it is equivalent to $|\arg f'/g'| < \pi/2$. This means that the variation of the argument of f' is limited by a convex function, providing the definition with a more concrete geometric motivation. For more information on close-to-convex functions, see [14, Chapter 2] and [56, Chapter 2].

We now turn our attention to the generalization of local univalence to harmonic functions. Let f be a complex-valued harmonic function in $\mathbb{D}$. Then f has the unique representation $f = h + \overline{g}$, where both h and g are analytic in $\mathbb{D}$ and $g(0) = 0$. In this case, f is orientation preserving and locally univalent, denoted by $f \in U_{\mathrm{loc}}^{H}$, if and only if its Jacobian $J_f = |h'|^2 - |g'|^2$ is strictly positive, by a result by Lewy [38]. In this case, $h \in U_{\mathrm{loc}}^{A}$ and the dilatation $\omega_f = \omega = g'/h'$ is analytic in $\mathbb{D}$ and maps $\mathbb{D}$ into itself. Clearly $f = h + \overline{g}$ is analytic if and only if the function g is constant.

We can calculate the pre-Schwarzian and Schwarzian derivatives for $f = h + \overline{g} \in U_{\mathrm{loc}}^{H}$ similarly to their analytic versions using their definitions including the Jacobian $J_f = |h'|^2 - |g'|^2$, resulting in

$$P_f = P_h - \frac{\overline{\omega}\omega'}{1 - |\omega|^2},$$

and

$$S_f = S_h + \frac{\overline{\omega}}{1 - |\omega|^2}\left(\frac{h''}{h'}\omega' - \omega''\right) - \frac{3}{2}\left(\frac{\overline{\omega}\omega'}{1 - |\omega|^2}\right)^2.$$

This generalization of P_f and S_f to harmonic functions was introduced and motivated in [29]. Extensions of univalence criteria for harmonic functions are further discussed in Paper **I**.

3 Function spaces

In this chapter, we present the fundamentals on some function spaces and classes that are relevant in the context of this thesis. We begin by discussing some properties of radial weights and proceed to both classical function spaces as well as their weighted versions induced by the aforementioned weights.

We adopt the following notation. If there exists a constant $C = C(\cdot) > 0$ for functions $a, b : I \to [0, \infty)$ defined on some set I such that $a(x) \leq Cb(x)$ for all $x \in I$, we write $a(x) \lesssim b(x)$ or $a \lesssim b$ for short. The converse $a \gtrsim b$ is defined analogously. If $a \lesssim b$ and $a \gtrsim b$, we write $a \asymp b$ to indicate that the functions a and b are comparable or asymptotically equivalent.

3.1 CLASSES OF WEIGHTS

Let D be a domain of the complex plane $\mathbb{C}$. An integrable function $\omega : D \to [0, \infty)$ is called a weight. Such a function induces a measure on the domain D, for which we write $\omega(E) = \int_E \omega(z)\, dA(z)$ for measurable subsets $E \subset D$. If $\omega(z) = \omega(|z|)$ for all $z \in D$, then ω is a radial weight. For the purposes of this thesis, we always assume $D = \mathbb{D}$ unless specified otherwise. We also assume that $\hat{\omega}(z) = \int_{|z|}^1 \omega(s) ds > 0$ for all $z \in \mathbb{D}$, for otherwise the weighted spaces of analytic functions considered in this overview would consist of all analytic functions. For each $x \in \mathbb{R}$ and a weight ν, we adopt the notation $\nu_{[\beta]}(z) = \nu(z)(1 - |z|^2)^\beta$. A radial weight ω belongs to the class of doubling weights $\widehat{\mathcal{D}}$ if there exists a constant $C = C(\omega) > 1$ such that

$$\hat{\omega}(r) \leq C\hat{\omega}\left(\frac{1+r}{2}\right), \quad 0 \leq r < 1.$$

Similarly, if there exist constants $K = K(\omega) > 1$ and $C = C(\omega) > 1$ such that

$$\hat{\omega}(r) \geq C\hat{\omega}\left(1 - \frac{1-r}{K}\right), \quad 0 \leq r < 1, \tag{3.1}$$

we write $\omega \in \check{\mathcal{D}}$. Observe that by a direct calculation (3.1) is equivalent to

$$\hat{\omega}(r) \leq \left(1 + \frac{1}{C-1}\right) \int_r^{1-\frac{1-r}{K}} \omega(t)\, dt, \quad 0 \leq r < 1.$$

The intersection $\widehat{\mathcal{D}} \cap \check{\mathcal{D}}$ is denoted by $\mathcal{D}$. The dual nature of the classes $\widehat{\mathcal{D}}$ and $\check{\mathcal{D}}$ can be seen via the known results stating that $\omega \in \widehat{\mathcal{D}}$ if and only if there exist $C = C(\omega) > 0$ and $\beta = \beta(\omega) > 0$ such that

$$\hat{\omega}(r) \leq C\left(\frac{1-r}{1-t}\right)^\beta \hat{\omega}(t), \quad 0 \leq r \leq t < 1, \tag{3.2}$$

see [46, Lemma 2.1]. In the same sense, it can be proven using similar arguments, see for example [47, Lemma B], that $\omega \in \check{\mathcal{D}}$ if and only if there exist $C = C(\omega) > 0$

and $\gamma = \gamma(\omega) > 0$ such that

$$\widehat{\omega}(t) \leq C \left(\frac{1-t}{1-r}\right)^{\gamma} \widehat{\omega}(r), \quad 0 \leq r \leq t < 1,$$

which is effectively the converse of (3.2). It follows that, for $0 \leq r < 1$ and a radial weight ω of the unit disc, $\widehat{\omega}(r)(1-r)^{-\beta}$ is essentially increasing for some $\beta > 0$ when $\omega \in \widehat{\mathcal{D}}$, and similarly $\widehat{\omega}(r)(1-r)^{-\gamma}$ is essentially decreasing and for some $\gamma > 0$ when $\omega \in \check{\mathcal{D}}$. This gives an insight to how the radial weights in classes $\widehat{\mathcal{D}}$ and $\check{\mathcal{D}}$ compare to the classical weights $\omega_\alpha(z) = (1-|z|^2)^\alpha$, $-1 < \alpha < \infty$, since clearly $1 - r \asymp 1 - r^2$ for $0 \leq r < 1$. Namely, weights in the class $\widehat{\mathcal{D}}$ cannot decay to zero too rapidly while weights in $\check{\mathcal{D}}$ cannot grow too fast when compared with the classical weight. Weights in the intersection $\mathcal{D}$ can be thought of as being closest to the classical radial weights. To give concrete examples, let $1 < \alpha < \infty$ and consider the weight v_α defined by

$$v_\alpha(z) = \frac{1}{(1-|z|)\left(\log \frac{e}{1-|z|}\right)^{\alpha}}, \quad z \in \mathbb{D}, \tag{3.3}$$

for which $\widehat{v_\alpha}(z) \asymp \left(\log \frac{e}{1-|z|}\right)^{1-\alpha}$ and $v_\alpha \in \widehat{\mathcal{D}} \setminus \mathcal{D}$. On the other hand, for the weight ω_γ defined by

$$\omega_\gamma(z) = \frac{\gamma}{(1-|z|)^2 e^{\frac{\gamma}{1-|z|}}}, \quad z \in \mathbb{D}, \quad \gamma > 0,$$

we have $\widehat{\omega_\gamma}(z) \asymp \exp\left(\frac{-\gamma}{1-|z|}\right)$ and $\omega_\gamma \in \check{\mathcal{D}} \setminus \mathcal{D}$, emphasizing the different natures of the two classes of weights. We proceed to provide useful characterizations for $\widehat{\mathcal{D}}$ and $\check{\mathcal{D}}$ later in the summary of Paper **III**.

There is a third class of weights of the unit disc which is of special interest in the context of this thesis. For $1 \leq x < \infty$ we define the moment ω_x of a weight ω as

$$\omega_x = \int_0^1 r^x \omega(r)\, dr, \quad 0 \leq r < 1.$$

Furthermore, if there exist $K = K(\omega) > 1$ and $C = C(\omega) > 1$ such that $\omega_x \geq C\omega_{Kx}$, then we write $\omega \in \mathcal{M}$. It is known that $\check{\mathcal{D}} \subsetneq \mathcal{M}$, the proof of which can be found in [50, Proposition 14], and $\mathcal{D} = \widehat{\mathcal{D}} \cap \check{\mathcal{D}} = \widehat{\mathcal{D}} \cap \mathcal{M}$, see [50, Theorem 3]. In addition, the following lemma shows a useful characterization for the class $\mathcal{M}$. This lemma originates from [50] wherefrom its proof can be found.

Lemma 3.1. *[50] Let ω be a radial weight. Then the following statements are equivalent:*

(i) $\omega \in \mathcal{M}$;

(ii) There exist $C = C(\omega) > 0$ and $K = K(\omega) > 1$ such that

$$\widehat{\omega}(t) \leq C \int_0^t s^{\frac{1}{K(1-t)}} \omega(s)\, ds, \quad 1 - \frac{1}{K} \leq t < 1;$$

(iii) For some (equivalently for each) $\beta > 0$, there exists $C = C(\omega, \beta) > 0$ such that

$$\omega_x \leq C x^\beta \left(\omega_{[\beta]}\right)_x, \quad 1 \leq x < \infty.$$

Lemma 3.1(ii) shows that $\mathcal{M}$ is closed under multiplication under any non-increasing weight. In particular, if $\omega \in \mathcal{M}$, then $\omega_{[\beta]} \in \mathcal{M}$ for each $\beta > 0$. For more information on the classes $\widehat{\mathcal{D}}$, $\check{\mathcal{D}}$, $\mathcal{D}$, $\mathcal{M}$ and other related classes, see [46,48,50] and the relevant references therein.

3.2 FUNCTION SPACES AND UNIVALENT FUNCTIONS

In this section, we present and discuss necessary function spaces to put the upcoming summaries of papers into their proper context. We also discuss some results regarding univalent functions in those spaces. Recall that $\mathcal{H}(\mathbb{D})$ is the set of all analytic functions in $\mathbb{D}$. Each weighted function space and class presented in this chapter reduces to its non-weighted version with the natural choice of $\omega \equiv 1$ and to the classically weighted version with $\omega_\alpha(z) = (1-|z|^2)^\alpha$, $\alpha > -1$. In the latter case, for a function space or class X, we write $X_{\omega_\alpha} = X_\alpha$.

3.2.1 Hardy, Bergman and Dirichlet spaces

For $f \in \mathcal{H}(\mathbb{D})$ the L^p-integral means M_p are defined via the restrictions of f to the circle of radius r, namely

$$M_p(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta}|^p \, d\theta\right)^{\frac{1}{p}}, \quad 0 < r < 1,$$

and we set $M_\infty(r,f) = \max_{|z|=r} |f(z)|$. For $0 < p < \infty$, the Hardy space H^p consists of all analytic functions $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{H^p} = \sup_{0<r<1} M_p(r,f) < \infty.$$

We also set

$$J^p_\omega(f) = \int_0^1 M^p_\infty(r,f)\omega(r)\, dr, \quad f \in \mathcal{H}(\mathbb{D}),$$

which will prove to be a useful notation in comparing certain kinds of Hardy and Dirichlet type norms with the maximum modulus. For the fundamental theory of H^p spaces, see the classical monograph [13]. The weighted Bergman space A^p_ω consists of functions $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|^p_{A^p_\omega} = \int_{\mathbb{D}} |f(z)|^p \omega(z)\, dA(z) < \infty,$$

where $\pi dA(z) = dx\, dy = d\theta\, rdr$, that is, dA is the Lebesgue area measure on the unit disc normalized such that $dA(\mathbb{D}) = 1$. For the theory of Bergman spaces, see [15,27]. The relation between the norms of H^p and A^p can be seen through the Hardy-Spencer-Stein identity

$$\|f\|^p_{H^p} = \frac{1}{2\pi}\int_{\mathbb{D}} \Delta|f|^p(z) \log\frac{1}{|z|}\, dA(z) + |f(0)|^p, \quad f \in \mathcal{H}(\mathbb{D}), \tag{3.4}$$

where $\Delta|f|^p = p^2|f|^{p-2}|f'|^2$ is the Laplacian of $|f|^p$, see [60] in relation to the Hardy-Spencer-Stein identity. It is well known that the Hardy space H^p is, roughly speaking, a limit case of the classically weighted Bergman space A^p_α as $\alpha \to -1^+$ [65]. We proceed to define

$$\|f\|_{H^p_\omega}^p = \int_0^1 \left(\int_{D(0,r)} \Delta|f|^p \, dA \right) \omega(r)\, dr = \int_{\mathbb{D}} \Delta|f|^p \widehat{\omega}\, dA, \quad f \in \mathcal{H}(\mathbb{D}).$$

Clearly H^p_ω coincides with the Hardy space H^p for $\omega \equiv 1$ by (3.4). We also set

$$\|f\|_{S^p_\omega}^p = \int_0^1 \left(\int_{D(0,r)} |f'|^2 \, dA \right)^{\frac{p}{2}} \omega(r)\, dr = \int_0^1 \mathrm{Area}(f(D(0,r)))^{\frac{p}{2}} \omega(r)\, dr, \quad f \in \mathcal{H}(\mathbb{D}),$$

where $\mathrm{Area}(f(D(0,r)))$ denotes the area of the image of $D(0,r)$ under f counting multiplicities, giving the integral quantity an obvious geometric meaning. We denote by H^p_ω and S^p_ω the spaces of analytic functions $f \in \mathcal{H}(\mathbb{D})$ for which the above respective integral quantities are finite. These spaces originate for the classical weights in [31, 42] and further appear in the study of conformal maps in [22, 31, 52].

For $0 < p < \infty$, the weighted Dirichlet space D^p_ω is defined by

$$\|f\|_{D^p_\omega}^p = \|f'\|_{A^p_\omega}^p + |f(0)|^p = \int_{\mathbb{D}} |f'(z)|^p \omega(z)\, dA(z) + |f(0)|^p < \infty, \quad f \in \mathcal{H}(\mathbb{D}).$$

The inclusions $D^p_{p-1} \subset H^p$ and $H^q \subset D^q_{q-1}$ for $0 < p \le 2 \le q < \infty$ are known due to results by Littlewood and Paley [40, Theorems 5 and 6]. The inclusions are strict whenever $p \neq 2$, see [4, pp. 839–840]. The univalent functions of the Hardy space and the classically weighted Dirichlet space have been studied in [4, 22, 23, 52]. Especially [4, Theorem 1] states that $H^p \cap \mathcal{U} = D^p_{p-1} \cap \mathcal{U}$, $0 < p < \infty$, showing that the univalent functions of H^p coincide with those of D^p_{p-1}.

The Hardy-Littlewood space HL_p consists of those $f \in \mathcal{H}(\mathbb{D})$ whose Maclaurin series coefficients satisfy $\|f\|_{\mathrm{HL}_p}^p = \sum_{k=0}^\infty |\widehat{f}(k)|^p (k+1)^{p-2} < \infty$. For background on Hardy-Littlewood spaces, see [13, pp. 95–98] and [26, 31]. Recently, a weighted version of HL_p naturally emerged in relation to the study of integration operators in [51]. There, for $0 < p < \infty$ and a radial weight ω, the weighted Hardy-Littlewood space HL_p^ω was defined by the condition

$$\|f\|_{\mathrm{HL}_p^\omega}^p = \sum_{k=0}^{\infty} |\widehat{f}(k)|^p (k+1)^{p-2} \omega_{kp+1} < \infty. \tag{3.5}$$

Since the Hardy-Littlewood inequalities [13, Theorems 6.2 and 6.3] provide that $H^p \subset \mathrm{HL}_p$ for $0 < p \le 2$ and $\mathrm{HL}_p \subset H^p$ for $2 \le p < \infty$, we have the inclusion chains

$$D^p_{p-1} \subset H^p \subset \mathrm{HL}_p, \quad 0 < p \le 2,$$

and

$$\mathrm{HL}_p \subset H^p \subset D^p_{p-1}, \quad 2 \le p < \infty.$$

Similarly with the here combined relations between the Hardy and Dirichlet spaces stated before, the inclusions are strict whenever $p \neq 2$.

Finally, we denote

$$I_{p,q,\omega}(f) = \int_0^1 M_q^p(r, f')(1-r)^{p\left(1-\frac{1}{q}\right)} \omega(r)\, dr + |f(0)|^p, \quad f \in \mathcal{H}(\mathbb{D}),$$

for $0 < p, q < \infty$, and similarly $I_{p,q}(f)$ for short when $\omega \equiv 1$. While this quantity is strictly speaking not used in the manner of a function space per se, the notation will prove useful in the studies within Paper **II** for $\omega \equiv 1$ and in Paper **III** for more general radial weights. Using this notation, we state the following known characterization for univalent functions of the Hardy space:

Theorem 3.1. *[23, Theorem 2] Suppose that $f \in \mathcal{U}$ and $\beta = \frac{1}{2} - \frac{1}{316}$. Then the following statements hold.*

(i) For $0 < p \le q < \infty$, $f \in H^p$ if and only if

$$I_{p,q}(f) = \int_0^1 M_q^p(r, f')(1-r)^{p\left(1-\frac{1}{q}\right)}\, dr + |f(0)|^p < \infty, \quad f \in \mathcal{U}. \tag{3.6}$$

(ii) If $0 < p < \infty$ and (3.6) holds for some q with $\frac{p}{p+1} < q < p$, then $f \in H^p$.

(iii) If $0 < p < \frac{1}{\beta}$ and $f \in \mathcal{H}^p$, then (3.6) holds for all q with $\frac{p}{p+1-p\beta} < q < p$.

The constant $\beta = \frac{1}{2} - \frac{1}{316}$ appearing in the theorem has its roots in an article by Baernstein [3]. This result will be discussed further in the summary of Paper **II**.

3.2.2 Bloch, Besov and other spaces

The Bloch space $\mathcal{B}$ consists of functions $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}} = \sup_{z \in \mathbb{D}} |f'(z)|(1-|z|^2) + |f(0)| < \infty, \quad f \in \mathcal{H}(\mathbb{D}).$$

The above quantity is called the Bloch norm. The little Bloch space $\mathcal{B}_0$ consists of analytic functions for which $|f'(z)|(1-|z|^2) \to 0$ as $|z| \to 1^-$. Both [2] and [20] are great sources for fundamentals on Bloch functions, with latter also containing a chapter on their relation to univalent functions.

For $1 < p < \infty$ the classical Besov space B^p consists of functions $f \in \mathcal{H}(\mathbb{D})$ such that the Besov seminorm is finite, that is,

$$\|f\|_{B^p}^p = \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2}\, dA(z) < \infty.$$

Both the Bloch and Besov spaces are conformally invariant in the sense that $\|f \circ \varphi_a\|_{\mathcal{B}} = \|f\|_{\mathcal{B}}$ and similarly for the Besov space. For a good reference on descriptions of analytic Besov spaces of the unit disc, see [64]. Let $\Omega \subset \mathbb{C}$ a simply connected domain, denote $d_\Omega(w) = \mathrm{dist}(w, \partial\Omega)$, $w \in \Omega$, and recall the geometric estimates (2.2) for univalent functions. Walsh extended this result to Besov spaces in 2000 [63] in the following manner.

Theorem 3.2. *[63, Theorem 1] Let $1 < p < \infty$ and let Ω be a simply connected proper subdomain of $\mathbb{C}$. Suppose further that $f \in \mathcal{U}$ and $f(\mathbb{D}) = \Omega$. Then $f \in B^p$ if and only if*

$$\int_\Omega d_\Omega(w)^{p-2}\, dA(w) < \infty.$$

The same publication continues to show another similar description for the Besov space, see [63, Theorem 2]. In general, for a (locally) univalent function f, it is typical to see some kind of a correspondence between the analytic properties of f, like inclusions to certain function spaces or norm estimates, and the geometric properties of either $f(\mathbb{D})$ or $f(\mathbb{T})$. For example, if $f \in \mathcal{B} \cap \mathcal{U}$, then $f(\mathbb{D})$ cannot contain any arbitrarily large discs by (2.2). In addition, [18] contains multiple interesting results tying certain smoothness conditions of $f(\mathbb{T})$ to the analytic properties of $f \in U^A_{\text{loc}}$. There exist numerous results of the same essence as the phenomenon is somewhat prevalent within the study of univalent functions, but some additional good examples can be found in [17] as well as the classical monograph [57].

Although the research articles of this thesis do not contain studies related to the following spaces and we will not process them in a detailed manner, it is noteworthy that multiple geometric and analytic properties of (locally) univalent functions belonging to the BMOA and VMOA spaces have also been a topic of interest within existing literature. The function space BMOA of analytic functions with bounded mean oscillation on $\mathbb{T}$ consists of functions $f \in H^2$ such that

$$\|f\|^2_{\text{BMOA}} = \sup_{\xi \in \mathbb{D}} \|f_\xi\|^2_{H^2} = \sup_{\xi \in \mathbb{D}} \frac{1}{2\pi} \int_{\mathbb{T}} |f(z) - f(\xi)|^2 \frac{1 - |\xi|^2}{|z - \xi|^2} \, |dz| < \infty, \quad z, \xi \in \mathbb{D},$$

where $f_\xi(z) = (f \circ \varphi_\xi)(z) - f(\xi)$ and φ_ξ is the automorphism of the unit disc. The VMOA space of analytic functions with vanishing mean oscillation similarly consists of functions $f \in H^2$ such that $\|f_\xi\|_{H^2} \to 0$ as $|\xi| \to 1^-$. Regarding the univalent functions of these spaces, it is known that $\mathcal{B} \cap \mathcal{U} = \text{BMOA} \cap \mathcal{U}$ and $\mathcal{B}_0 \cap \mathcal{U} = \text{VMOA} \cap \mathcal{U}$, see [17, pp. 3] and the references therein. For further references regarding the study of univalent functions in these spaces, see [12, 17, 18, 21].

To conclude, we note that there is a certain expression that also turns out to play a significant role in the study of univalent functions in the spaces of analytic functions discussed in this subsection. Namely, the primitive of the pre-Schwarzian $\log f'$ expectedly appears in numerous results, especially in ones concerning the Bloch space $\mathcal{B}$ or BMOA. Typically, it should not be surprising to see some kind of results utilizing this expression when discussing univalent functions of function spaces that are defined in terms of the derivative of f. For example, majority of the results in [18] are formulated around this expression. In addition, some interesting results concerning the relation of Bloch and Dirichlet norms of $\log f'$ and the univalence of f can be found in [7].

4 Summary of papers

4.1 SUMMARY OF PAPER I

In Paper **I**, we consider conditions for the univalence of locally univalent analytic functions $f \in U_{\text{loc}}^A$ of the unit disc $\mathbb{D}$ in terms of their pre-Schwarzian derivative. In particular, we study functions restricted by the condition in (2.8). We find that imposing this condition guarantees the univalence of a function $f \in U_{\text{loc}}^A$ in certain horodiscs, that is, Euclidean discs $D(a,r) \subset \mathbb{D}$ internally tangent to the unit circle. We also consider as growth restrictions for the pre-Schwarzian in similar horodiscs. In addition, we derive an extension to distortion theorems found in classical research of univalent functions and touch upon generalizations of our main results to locally univalent harmonic functions.

4.1.1 Distortion theorems

Our first result can be seen as an extension to existing distortion theorems. By using arguments similar to those in the proof of [7, Theorem 3.2] and in [35], we obtain the following result.

Theorem 4.1. *[33, Theorem 1] Let f be meromorphic in $\mathbb{D}$ such that*

$$\left|\frac{f''(z)}{f'(z)}\right| \leq \varphi(|z|), \quad 0 \leq R \leq |z| < 1, \tag{4.1}$$

for some $\varphi : [R, 1) \to [0, \infty)$.

(i) If

$$\limsup_{r \to 1^-} (1-r) \exp\left(\int_R^r \varphi(t)\, dt\right) < \infty, \tag{4.2}$$

then $\sup\limits_{R<|z|<1} |f'(z)|(1-|z|^2) < \infty.$

(ii) If

$$\int_R^1 \exp\left(\int_R^s \varphi(t)\, dt\right) ds < \infty, \tag{4.3}$$

then $\sup\limits_{R<|z|<1} |f(z)| < \infty.$

Clearly the required bound in (4.1) for the pre-Schwarzian reduces to (2.1) by choosing $\varphi(t) = (4+2t)/(1-t^2)$. The assumptions (i) and (ii) in Theorem 4.1 are respectively satisfied by the functions

$$\varphi(t) = \frac{2}{1-t^2} = \left(\log \frac{1+t}{1-t}\right)'$$

and

$$\psi(t) = \frac{B}{1-t^2} + \frac{C}{1-t^2}\left(\log\frac{e}{1-t}\right)^{-(1+\varepsilon)},$$

where $\varepsilon, C > 0$ and $0 < B < 2$. By Theorem 4.1, if f is meromorphic in $\mathbb{D}$ and satisfies (4.1) and (4.2) for some φ, then $f \in \mathcal{N}$. Moreover, if f is also analytic in $\mathbb{D}$, then $f \in \mathcal{B}$, and if (4.3) holds, then f is bounded.

4.1.2 Main results

Our first main result concerns Becker's univalence criterion in a neighborhood of a boundary point $\zeta \in \mathbb{T}$.

Theorem 4.2. *[33, Theorem 2] Let $f \in U_{loc}^A$ and $\zeta \in \mathbb{T}$. If there exists a sequence $\{w_n\}$ of points in $\mathbb{D}$ tending to ζ such that*

$$\left|\frac{f''(w_n)}{f'(w_n)}\right|(1-|w_n|^2) \to c \tag{4.4}$$

for some $6 < c \leq \infty$, then for each $\delta > 0$ there exists a point $w \in f(\mathbb{D})$ such that at least two of its distinct preimages belong to $D(\zeta,\delta) \cap \mathbb{D}$.

Conversely, if for each $\delta > 0$ there exists a point $w \in f(\mathbb{D})$ such that at least two of its distinct preimages belong to $D(\zeta,\delta) \cap \mathbb{D}$, then there exists a sequence $\{w_n\}$ of points in $\mathbb{D}$ tending to ζ such that (4.4) *holds for some $1 \leq c \leq \infty$.*

It is clear that (4.4) does not guarantee infinite valence for f when $c > 6$. For example, the polynomial $f(z) = (1-z)^{2n+1}$, $n \in \mathbb{N}$, satisfies the sharp inequality

$$\left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2) \leq 4n, \quad z \in \mathbb{D},$$

although $f(z) = \varepsilon^{2n+1}$ has n solutions in $D(1,\delta) \cap \mathbb{D}$ for each $0 < \varepsilon < \delta$ when $0 < \delta(n) < 1$ is small enough. Similarly, the function $f(z) = (1-z)^{-p}$, $0 < p < \infty$, satisfies the sharp inequality

$$\left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2) \leq 2(p+1), \quad z \in \mathbb{D},$$

and for each $2n < p \leq 2n+2$, $n \in \mathbb{N} \cup \{0\}$, the valence of f is $n+1$ for suitably chosen points in the image set.

Let $0 < a < 1$ and $0 \leq \theta < 2\pi$. Then the function $T(z) = ae^{i\theta} + (1-a)z$, $z \in \mathbb{D}$, maps the unit disc $\mathbb{D}$ onto the horodisc $D(ae^{i\theta}, 1-a)$. We proceed to use this localization to study the behaviour of locally analytic functions in said horodiscs by considering the composed mapping $g = f \circ T$. The following result shows that condition (2.8) guarantees univalence in horodiscs of the unit disc for $f \in U_{\mathrm{loc}}^A$.

Theorem 4.3. *[33, Theorem 4] Let $f \in U_{loc}^A$ and assume that* (2.8) *holds for some $0 < C < \infty$. If $0 < C \leq 1$, then f is univalent in $\mathbb{D}$. If $1 < C < \infty$, then there exists $0 < a < 1$, $a = a(C)$, such that f is univalent in all discs $D(ae^{i\theta}, 1-a)$, $0 \leq \theta < 2\pi$. In particular, we can choose $a = 1-(1+C)^{-2}$.*

Conversely, if f is univalent in all horodiscs of the unit disc, we can derive an estimate for the growth of its valence. Let $f \in U^A_{\text{loc}}$ be univalent in each horodisc $D(ae^{i\theta}, 1-a)$, $0 \le \theta < 2\pi$, for some $0 < a < 1$. By the proof of [24, Theorem 6], for each $w \in f(\mathbb{D})$, the sequence of pre-images $\{z_n\} \in f^{-1}(w)$ satisfies

$$\sum_{z_n \in Q} (1-|z_n|)^{1/2} \le K\ell(Q)^{1/2} \tag{4.5}$$

for any Carleson square Q and some constant $0 < K < \infty$ depending on a. Here

$$Q = Q(I) = \left\{ re^{i\theta} \,:\, e^{i\theta} \in I,\ 1 - \frac{|I|}{2\pi} \le r < 1 \right\}$$

is a Carleson square based on the arc $I \subset \partial\mathbb{D}$ and $|I| = \ell(Q)$ is the Euclidean arc length of I. By choosing $Q = \mathbb{D}$ in (4.5), we obtain

$$n(f, r, w) \lesssim \frac{1}{\sqrt{1-r}}, \quad r \to 1^-,$$

where $n(f, r, w)$ is the number of pre-images $\{z_n\} = f^{-1}(w)$ in the disc $\overline{D(0, r)}$. Namely, arrange $\{z_n\} = f^{-1}(w)$ by increasing modulus, and let $0 < |z_n| = r < |z_{n+1}|$ to deduce

$$(1-r)^{1/2} n(f, r, w) \le \sum_{k=0}^{n} (1-|z_k|)^{1/2} \le K\ell(\mathbb{D})^{1/2} < \infty$$

for some $0 < K(a) < \infty$.

We continue to show that, for $f \in U^A_{\text{loc}}$, univalence in certain horodiscs guarantees growth restrictions for its pre-Schwarzian derivative. In the next theorem we consider slightly larger horodiscs in comparison to Theorem 4.3 by choosing $a = 1 - (1+C)^{-1}$.

Theorem 4.4. *[33, Theorem 5] Let $f \in U^A_{loc}$ be univalent in all Euclidean discs*

$$D\left(\frac{C}{1+C} e^{i\theta}, \frac{1}{1+C}\right), \quad e^{i\theta} \in \partial\mathbb{D},$$

for some $0 < C < \infty$. Then

$$\left|\frac{f''(z)}{f'(z)}\right| (1-|z|^2) \le 2 + 4(1 + K(z)), \quad z \in \mathbb{D},$$

where $K(z) \asymp (1-|z|^2)$ as $|z| \to 1^-$.

In view of (2.1), Theorem 4.4 is in a sense sharp. Moreover, since (2.1) implies

$$\left|\frac{f''(z)}{f'(z)}\right| (1-|z|) \le \frac{4+2|z|}{1+|z|} \le 4$$

for univalent analytic functions f, the last main result of this paper is sharp as well and is essentially a direct consequence of the above estimate.

Theorem 4.5. *[33, Theorem 6] Let $f \in U^A_{loc}$ be univalent in all Euclidean discs*

$$D(ae^{i\theta}, 1-a) \subset \mathbb{D}, \quad e^{i\theta} \in \partial\mathbb{D},$$

for some $0 < a < 1$. Then

$$\left|\frac{f''(z)}{f'(z)}\right| (1-|z|) \le 4, \quad a \le |z| < 1.$$

4.1.3 Generalizations for harmonic functions

To conclude, we consider extensions of our main results to harmonic functions. Univalence criteria, analogous to those of Nehari's and Becker's, for harmonic functions have recently been discovered using the generalizations of the pre-Schwarzian and Schwarzian derivatives presented in Section 2.3. Namely, there exists $0 < \delta_0 < 2$ such that if $f \in U^H_{\text{loc}}$ satisfies (2.4) for $\delta = \delta_0$, then f is univalent in $\mathbb{D}$, see [1] and [28]. The sharp value of δ_0 is not known. Moreover, if $f \in U^H_{\text{loc}}$ satisfies

$$|P_f(z)|(1-|z|^2) + \frac{|\omega'(z)|(1-|z|^2)}{1-|\omega(z)|^2} \leq 1, \quad z \in \mathbb{D},$$

then f is univalent. The constant 1 is sharp, by the sharpness of Becker's univalence criterion. If one of these mentioned inequalities holds in an annulus $r_0 < |z| < 1$ for some $0 < r_0 < 1$ with a slightly smaller right-hand-side constant, then f is of finite valence [32]. Conversely to these univalence criteria, there exist absolute constants $0 < C_1, C_2 < \infty$ such that if $f \in U^H_{\text{loc}}$ is univalent, then (2.4) holds for $\delta = C_1$ and (2.7) holds for $\rho = C_2$, see [29]. The sharp values of C_1 and C_2 are not known.

By the above-mentioned analogues of Nehari's criterion, Becker's criterion and their converses, we obtain generalizations of the results in this paper for harmonic functions. Of course, the correct operators and constants have to be involved. Theorem 4.2 and its analogue [24, Theorem 1] for the Schwarzian derivative S_f are valid as well. Moreover, the generalizations of Theorems 4.3, 4.4, and 4.5 to harmonic mappings are valid.

Finally, we state an important generalization of [24, Theorem 3] for harmonic functions. It gives a sufficient condition for the Schwarzian derivative of $f \in U^H_{\text{loc}}$ such that the preimages of each $w \in f(\mathbb{D})$ are separated in the hyperbolic metric. Here $\xi(z_1, z_2)$ is the hyperbolic midpoint of the hyperbolic segment $\langle z_1, z_2 \rangle$ for $z_1, z_2 \in \mathbb{D}$.

Theorem 4.6. *[33, Theorem 9] Let $f = h + \overline{g} \in U^H_{loc}$ such that*

$$|S_H(f)|(1-|z|^2) \leq \delta_0(1 + C(1-|z|)), \quad z \in \mathbb{D},$$

for some $0 < C < \infty$. Then each pair of points $z_1, z_2 \in \mathbb{D}$ such that $f(z_1) = f(z_2)$ and $1 - |\xi(z_1, z_2)| < 1/C$ satisfies

$$d_H(z_1, z_2) \geq \log \frac{2 - C^{1/2}(1-|\xi(z_1, z_2)|)^{1/2}}{C^{1/2}(1-|\xi(z_1, z_2)|)^{1/2}}. \tag{4.6}$$

Conversely, if there exists a constant $0 < C < \infty$ such that each pair of points $z_1, z_2 \in \mathbb{D}$ for which $f(z_1) = f(z_2)$ and $1 - |\xi(z_1, z_2)| < 1/C$ satisfies (4.6), *then*

$$|S_H(f)|(1-|z|^2) \leq C_2(1 + \Psi_C(|z|)(1-|z|)^{1/3}), \quad 1 - |z| < (8C)^{-1},$$

where Ψ_C is positive, and satisfies $\Psi_C(|z|) \to (2(8C)^{1/3})^+$ as $|z| \to 1^-$.

4.2 SUMMARY OF PAPER II

In paper **II** we consider a known characterization of univalent functions of the Hardy space H^p in terms of an integral quantity over the L^p means of their derivatives. Namely, by combining [4, Theorem 1] with Theorem 3.1 we see that (3.6) holds if either $0 < p \leq q < \infty$ or $\frac{p}{1+p} < q < p < 2 + \frac{2}{157}$, where the last upper bound comes from the constant $\beta = \frac{1}{2} - \frac{1}{316}$ in Theorem 3.1. Our main result shows that these restrictions on p and q can be significantly loosened and that the estimate is, for certain values of p and q, valid for all close-to-convex functions.

Theorem 4.7. *[53, Theorem 1] Let $0 < p, q < \infty$ such that either $\frac{2p}{2+p} < q < 2$ or $q \geq 2$. Then*

$$\|f\|_{H^p}^p \asymp \int_0^1 M_q^p(r, f')(1-r)^{p\left(1-\frac{1}{q}\right)}\, dr + |f(0)|^p \tag{4.7}$$

for all $f \in \mathcal{U}$. Moreover, if $0 < p < \infty$ and $1 \leq q < \infty$, then (4.7) *is valid for all close-to-convex functions f.*

On one hand, Theorem 4.7 shows that for $q \geq 2$ there is no restriction on p. On the other hand, $\frac{2p}{2+p} \in (0, 2)$ for all $0 < p < \infty$, and hence the range $\frac{2p}{2+p} < q < 2$ covers many cases previously excluded by the requirement $p < 2 + \frac{2}{157}$. However, the hypothesis $\frac{2p}{2+p} < q$ is obviously strictly stronger than $\frac{p}{1+p} < q$ for each $0 < p < \infty$. The statement on close-to-convex functions is a generalization of [23, Proposition 1] concerning the case $q = 1$. In particular, we offer two proofs concerning the case $q \geq 2$, one of which reveals that for $p \geq q$ we have an asymptotic equality which we have previously not found in the literature and consider to be of interest, namely

$$\|f\|_{H^p}^p \asymp \int_0^1 \left(\int_{D(0,r)} \Delta |f'|^q(z)\, dA(z)\right)^{\frac{p}{q}} (1-r)^p\, dr, \quad f \in \mathcal{U}.$$

We proceed to show that Theorem 4.7 has an extension to the weighted Bergman space. The natural approach we adopt to obtain (4.8) consists of first applying (4.7) to the univalent dilatation $f_r(z) = f(rz)$ appearing in the Bergman space norm of f, and then changing the order of radial integrations. The problem then no longer involves the weight ω and the final step is managed by using the well-known fact that, for $f \in \mathcal{U}$, the estimate $|f'(\rho\xi)| \asymp |f'(r\xi)|$ holds for all $\xi \in \mathbb{T}$ and $0 \leq r \leq \rho < 1$ such that $1 - r \asymp 1 - \rho$, see for example [57, Corollary 1.6].

Corollary 4.1. *[53, Corollary 2] Let $0 < p, q < \infty$ and let $\omega : \mathbb{D} \to [0, \infty)$ such that $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$. Further, assume that one of the following conditions is satisfied:*

(i) $0 < p \leq q < \infty$;

(ii) $\frac{p}{1+p} < q < p < 2 + \frac{2}{157}$;

(iii) $q \geq 2$;

(iv) $\frac{2p}{2+p} < q < 2$.

Then

$$\begin{aligned} \|f\|_{A_\omega^p}^p \asymp &\int_0^1 M_q^p(r, f')(1-r)^{p\left(1-\frac{1}{q}\right)} \left(\int_r^1 \omega(t) t\, dt\right) dr \\ &+ |f(0)|^p \int_0^1 \omega(r) r\, dr, \quad f \in \mathcal{U}. \end{aligned} \tag{4.8}$$

The case $q = p$ of Corollary 4.1 is of special interest. It states that

$$\|f\|_{A^p_\omega}^p \asymp \int_{\mathbb{D}} |f'(z)|^p (1-|z|)^{p-1} \left(\int_{|z|}^1 \omega(r) r \, dr \right) dA(z) + |f(0)|^p \int_0^1 \omega(r) r \, dr, \quad f \in \mathcal{U}.$$

It is well known that this asymptotic equality is valid for all $f \in \mathcal{H}(\mathbb{D})$ if ω is the standard radial weight $(1-|z|^2)^\alpha$ with $-1 < \alpha < \infty$. These kind of asymptotic equalities are known as Littlewood-Paley formulas. The rough idea behind these asymptotics is that f' behaves in a somewhat similar way as f divided by the distance from the boundary. However, it is known that all Bergman spaces do not admit this property. Namely, there exist radial weights ω such that

$$\|f\|_{A^p_\omega}^p \asymp \int_{\mathbb{D}} |f'(z)|^p (1-|z|)^p W(z) \, dA(z) + |f(0)|^p, \quad f \in \mathcal{H}(\mathbb{D}),$$

fails to be true for each non-negative radial function W on $\mathbb{D}$ unless $p = 2$ [48, Proposition 4.3].

4.3 SUMMARY OF PAPER III

In Paper **III** we consider a variety of inequalities related to Bergman and Dirichlet type spaces induced by radial weights, as well as some other function spaces. Some of the results obtained can be considered as generalizations of certain known special cases, especially ones known from existing literature consisting of similar analysis conducted on classically weighted function spaces, while others are completely new. Furthermore, we apply the obtained norm inequalities in order to relate the growth of the maximum modulus of a conformal map f, measured in terms of a weighted integrability condition, to a geometric quantity involving the area of image under f of a disc centered at the origin. Our findings in this direction yield geometric characterizations of conformal maps in certain weighted Dirichlet and Besov spaces.

4.3.1 Lemmas on radial weights

We begin by presenting necessary lemmas for the radial weights we proceed to use in the proofs of this paper, focusing on the classes $\widehat{\mathcal{D}}$, $\check{\mathcal{D}}$, $\mathcal{D}$ and $\mathcal{M}$ introduced earlier. While some of the weight-related lemmas are slight extensions on existing characterizations of weights of the aforementioned classes, others are unpublished results of J. A. Peláez and the paper's second author J. Rättyä, in which case the connection is explicitly stated. The first lemma contains useful characterizations of the class $\widehat{\mathcal{D}}$. For a proof of the fact that (i)–(iv) are equivalent, see [46, Lemma 2.1] and [50]. The last part of the lemma is new. It is worth noticing that it fails in the case $\beta = 0$ because by [49, Theorem 3] there exists a weight $\omega \notin \widehat{\mathcal{D}}$ such that $\widehat{\omega}_{[-1]} \in \widehat{\mathcal{D}}$. However, the proof shows that if $\omega \in \widehat{\mathcal{D}}$ then $\widehat{\omega}_{[\beta-1]} \in \widehat{\mathcal{D}}$ for each $-1 < \beta < \infty$, provided $\widehat{\omega}_{[\beta-1]} \in L^1$.

Lemma 4.1. *Let ω be a radial weight. Then the following statements are equivalent:*

(i) $\omega \in \widehat{\mathcal{D}}$;

(ii) There exist $C = C(\omega) \geq 1$ and $\beta = \beta(\omega) > 0$ such that

$$\frac{\widehat{\omega}(r)}{(1-r)^\beta} \leq C \frac{\widehat{\omega}(t)}{(1-t)^\beta}, \quad 0 \leq r \leq t < 1;$$

(iii) *For some (equivalently for each) $\beta > 0$ there exists a constant $C = C(\omega, \beta) > 0$ such that*

$$x^\beta \left(\omega_{[\beta]}\right)_x \le C\omega_x, \quad 0 \le x < \infty;$$

(iv) *There exists $C = C(\omega) > 0$ such that $\omega_x \le C\omega_{2x}$ for all $0 \le x < \infty$;*

(v) *$\widehat{\omega}_{[\beta-1]} \in \widehat{\mathcal{D}}$ for some (equivalently for each) $\beta > 0$.*

The next elementary lemma proves to be useful for studying the types of inequalities examined in this paper. The special case $q = p$ shows that $\widehat{\omega} \lesssim \widehat{\nu}$ on $[0,1)$ is a sufficient condition for the identity operator $I : A^p_\nu \to A^p_\omega$ to be bounded.

Lemma 4.2. *[54, Lemma 8] Let $0 < p < \infty$, $0 < q \le \infty$ and $0 \le \rho < 1$, and let ω and ν be radial weights such that $\widehat{\omega} \lesssim \widehat{\nu}$ on $[\rho, 1)$. Then*

$$\int_\rho^1 M_q^p(r,f)\omega(r)\,dr \lesssim \int_\rho^1 M_q^p(r,f)\nu(r)\,dr, \quad f \in \mathcal{H}(\mathbb{D}).$$

A basic result that we will need is a set of characterizations of weights in $\check{\mathcal{D}}$ given in the next lemma. The characterization (ii) is well known and a detailed proof can be found in [47], while the characterizations (iii)-(vii) are unpublished results by J. A. Peláez and the second author J. Rättyä, and (viii) is new. In this paper we do not use the conditions (iii) and (iv) as such but as the proof passes naturally through them and the characterizations seem useful they are included here for the convenience of the reader and for further reference. The points $\rho_n = \rho_n(\omega, K) \in [0,1)$ appearing in (iv) are defined by the identity

$$\rho_n = \rho_n(\omega, K) = \min\{0 \le r < 1 : \widehat{\omega}(r) = \widehat{\omega}(0)K^{-n}\}, \quad 1 < K < \infty, \quad n \in \mathbb{N} \cup \{0\}.$$

Observe that $\rho_0 = 0 < \rho_1 < \cdots < \rho_n < \rho_{n+1} < \cdots$ for all $n \in \mathbb{N} \setminus \{1\}$, and $\rho_n \to 1^-$ as $n \to \infty$.

Lemma 4.3. *Let ω be a radial weight. Then the following statements are equivalent:*

(i) $\omega \in \check{\mathcal{D}}$;

(ii) *There exist $C = C(\omega) > 0$ and $\beta = \beta(\omega) > 0$ such that*

$$\frac{\widehat{\omega}(t)}{(1-t)^\beta} \le C\frac{\widehat{\omega}(r)}{(1-r)^\beta}, \quad 0 \le r \le t < 1;$$

(iii) *For some (equivalently for each) $\gamma \in (0, \infty)$, there exists $C = C(\gamma, \omega) > 0$ such that*

$$\int_0^r \frac{dt}{\widehat{\omega}(t)^\gamma (1-t)} \le \frac{C}{\widehat{\omega}(r)^\gamma}, \quad 0 \le r < 1;$$

(iv) *For some (equivalently for each) $K > 1$, there exists $C = C(\omega, K) > 0$ such that $1 - \rho_n \le C(1 - \rho_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$;*

(v) *For some (equivalently for each) $\beta \in (0, \infty)$, there exists $C = C(\beta, \omega) > 1$ such that*

$$\widehat{\omega}(r) \le C\frac{\widehat{\omega_{[\beta]}}(r)}{(1-r)^\beta}, \quad 0 \le r < 1;$$

(vi) *For some (equivalently for each) $\beta \in (0,\infty)$, there exists $C = C(\beta,\omega) \in (0,1)$ such that*

$$\frac{\int_r^1 \widehat{\omega}(t)\beta(1-t)^{\beta-1}\,dt}{(1-r)^\beta} \le C\widehat{\omega}(r), \quad 0 \le r < 1;$$

(vii) *For some (equivalently for each) $\gamma \in (0,\infty)$, there exists $C = C(\gamma,\omega) > 0$ such that*

$$\int_r^1 \frac{\widehat{\omega}(s)^\gamma}{1-s}\,ds \le C\widehat{\omega}(r)^\gamma, \quad 0 \le r < 1;$$

(viii) $\widehat{\omega}_{[\beta-1]} \in \check{\mathcal{D}}$ *for some (equivalently for each)* $0 \le \beta < \infty$.

We proceed to prove a proposition which provides us with more necessary tools to handle the main results of our paper. Although (i) is an estimate concerning Bergman space norms for all analytic functions, the essence of this result lies in handling the weights in an appropriate manner to gain all three parts of the proposition.

Proposition 4.1. *[54, Proposition 10] Let ω be a radial weight and $0 < p < \infty$. Then the following statements hold:*

(i) *If $0 < q < 1$, then*

$$\|f\|_{A^p_{\widehat{\omega}_{[q-2]}}} \lesssim \|f\|_{A^p_{\omega_{[q-1]}}}, \quad f \in \mathcal{H}(\mathbb{D}); \tag{4.9}$$

(ii) *If $1 \le q < \infty$, then a necessary condition for* (4.9) *to hold is that both $\widehat{\omega}_{[q-2]}$ and ω belong to $\mathcal{M}$;*

(iii) *If $1 \le q < \infty$, then a sufficient condition for* (4.9) *to hold is $\omega \in \check{\mathcal{D}}$.*

The next lemma is an unpublished result by J. A. Peláez and the second author J. Rättyä. It contains a set of characterizations of the class $\mathcal{M}$, and it should be compared with Lemma 3.1 in Section 3.1.

Lemma 4.4. *Let ω be a radial weight. Then the following statements are equivalent:*

(i) $\omega \in \mathcal{M}$;

(ii) *For some (equivalently for each) $0 < \gamma < \infty$, there exists $C = C(\omega,\gamma) > 0$ such that*

$$\int_x^\infty \omega_y^\gamma \frac{dy}{y} \le C\omega_x^\gamma, \quad 1 \le x < \infty;$$

(iii) *For some (equivalently for each) $0 \le \beta < \infty$, there exists $C = C(\omega,\beta) > 0$ such that*

$$\int_x^\infty \omega_y \frac{dy}{y^{\beta+1}} \le C\frac{\omega_x}{x^\beta}, \quad 1 \le x < \infty;$$

(iv) *For some (equivalently for each) $0 \le \beta < \infty$, there exists $C = C(\omega,\beta) > 0$ such that*

$$\int_{1-\frac{1}{x}}^1 \widehat{\omega}_{[\beta-1]}(t)\,dt \le C\frac{\omega_x}{x^\beta}, \quad 1 \le x < \infty.$$

We will need one more auxiliary result concerning the class $\mathcal{M}$. It is the next lemma which is an unpublished result by J. A. Peláez and the second author J. Rättyä.

Lemma 4.5. *[54, Lemma 16] Let $\omega \in \mathcal{M}$. Then there exists $\beta = \beta(\omega) > 0$ such that $\omega_{[-\beta]} \in \mathcal{M}$.*

4.3.2 Auxiliary results

We continue to present the auxiliary results and lemmas used to obtain our main results. Many of these lemmas prove more than what is strictly needed for the asymptotic estimates in our main results alone. The first lemma considers different ways of attaining upper bounds for J^p_ω in terms of suitable Dirichlet space norms.

Proposition 4.2. *[54, Proposition 12] Let ω be a radial weight. Then the following statements hold:*

(i) If $0 < p \le 1$, then $J^p_\omega(f) \lesssim \|f'\|^p_{A^p_{\widehat{\omega}_{[p-2]}}}$ for all $f \in \mathcal{S}$;

(ii) If $0 < p < 1$ and $\omega \in \widehat{\mathcal{D}}$, then $J^p_\omega(f) \lesssim \|f'\|^p_{A^p_{\widehat{\omega}_{[p-2]}}}$ for all $f \in \mathcal{H}(\mathbb{D})$;

(iii) $J^1_\omega(f) \lesssim J^1_{\widehat{\omega}}(f') + |f(0)| \lesssim \|f'\|_{A^1_\omega} + |f(0)|$ for all $f \in \mathcal{H}(\mathbb{D})$;

(iv) If $1 < p < \infty$ and $\omega \in \check{\mathcal{D}}$, then

$$J^p_\omega(f) \lesssim \|f'\|^p_{A^p_{\omega_{[p-1]}}} + |f(0)|^p, \quad f \in \mathcal{H}(\mathbb{D}).$$

Moreover, this estimate is in general false for the class $\mathcal{S}$ if $\omega \in \widehat{\mathcal{D}} \setminus \mathcal{D}$.

In the next result, we find upper bounds for $\|f\|^p_{H^p_\omega}$ and $\|f\|^p_{S^p_\omega}$ for univalent functions in terms of $J^p_\omega(f)$.

Lemma 4.6. *[54, Lemma 13] Let $0 < p < \infty$ and ω a radial weight. Then*

$$\|f\|^p_{H^p_\omega} \le 2\pi p J^p_\omega(f), \quad f \in \mathcal{U},$$

and

$$\|f\|^p_{S^p_\omega} \le \pi^{\frac{p}{2}} J^p_\omega(f), \quad f \in \mathcal{U}.$$

None of the corresponding asymptotic inequalities is valid for all $f \in \mathcal{H}(\mathbb{D})$ unless ω vanishes almost everywhere on $\mathbb{D}$.

The last assertion on the necessity of univalence of f can be deduced with the following reasoning. The monomial $m_n(z) = z^n$, $n \in \mathbb{N}$, satisfies

$$J^p_\omega(m_n) = \omega_{np}, \quad \|m_n\|^p_{H^p_\omega} = 2\pi n p \omega_{np}, \quad \text{and} \quad \|m_n\|^p_{S^p_\omega} = (\pi n)^{\frac{p}{2}} \omega_{np}, \quad n \in \mathbb{N},$$

and hence $J^p_\omega(m_n)$ does not dominate $\|m_n\|^p_{H^p_\omega}$ nor $\|m_n\|^p_{S^p_\omega}$ unless ω vanishes almost everywhere on $\mathbb{D}$.

For $f \in \mathcal{H}(\mathbb{D})$ with the Maclaurin series expansion $f(z) = \sum_{k=0}^\infty \widehat{f}(k) z^k$, write

$$P(r, f) = \sum_{k=1}^\infty |\widehat{f}(k)| r^k, \quad 0 \le r < 1.$$

We proceed to state a generalization of [25, Theorem 15] and [42, Proposition 2] to doubling weights. We show that the quantity J^p_ω is bounded by $\|f\|^p_{S^p_\omega}$ for suitably chosen radial weights.

Lemma 4.7. *[54, Lemma 14] Let $0 < p < \infty$ and $\omega \in \mathcal{D}$. Then*

$$J_\omega^p(f) \lesssim \int_0^1 P(r,f)^p \omega(r)\,dr + |f(0)|^p \lesssim \|f\|_{S_\omega^p}^p + |f(0)|^p, \quad f \in \mathcal{H}(\mathbb{D}).$$

Moreover, the estimate $J_\omega^p(f) \gtrsim \|f\|_{S_\omega^p}^p + |f(0)|^p$ is in general false for the class $\mathcal{S}$ if $0 < p < \infty$ and $\omega \in \widehat{\mathcal{D}} \setminus \mathcal{D}$.

The next result is a generalization of a known result based on [31, Theorem 1] and [42, Theorem 2] which state, that the estimates presented below are valid for the classical weight $\omega_\alpha(z) = (1-|z|^2)^\alpha$, $\alpha > -1$.

Lemma 4.8. *[54, Lemma 15] Let $\omega \in \mathcal{D}$. Then the following statements hold:*

(i) If $0 < p \leq 2$, then $\|f\|_{S_\omega^p} \lesssim \|f\|_{H_\omega^p}$ for all $f \in \mathcal{H}(\mathbb{D})$;

(ii) If $2 \leq p < \infty$, then $\|f\|_{H_\omega^p} \lesssim \|f\|_{S_\omega^p}$ for all $f \in \mathcal{H}(\mathbb{D})$.

Both estimates are in general false for the class $\mathcal{S}$ if $p \neq 2$ and $\omega \in \widehat{\mathcal{D}} \setminus \mathcal{D}$.

We continue with two lemmas allowing us to compare the functions in S_ω^p and the maximum modulus J_ω^p with Hardy-Littlewood type sums.

Lemma 4.9. *[54, Lemma 17] Let $\omega \in \mathcal{M}$. Then the following statements hold:*

(i) If $0 < p \leq 2$, then $\sum_{k=1}^\infty |\widehat{f}(k)|^p k^{p-1} \omega_{2k} \lesssim \|f\|_{S_\omega^p}^p$ for all $f \in \mathcal{H}(\mathbb{D})$;

(ii) If $2 \leq p < \infty$, then $\|f\|_{S_\omega^p}^p \lesssim \sum_{k=1}^\infty |\widehat{f}(k)|^p k^{p-1} \omega_{2k}$ for all $f \in \mathcal{H}(\mathbb{D})$.

Both estimates are in general false for the class $\mathcal{S}$ if $p \neq 2$ and $\omega \in \widehat{\mathcal{D}} \setminus \mathcal{D}$.

Lemma 4.10. *[54, Lemma 18] Let $1 < p < \infty$ and $\omega \in \mathcal{M}$. Then*

$$J_\omega^p(f) \lesssim \sum_{k=0}^\infty |\widehat{f}(k)|^p (k+1)^{p-1} \omega_k, \quad f \in \mathcal{H}(\mathbb{D}).$$

This estimate is in general false for the class $\mathcal{S}$ if $\omega \in \widehat{\mathcal{D}} \setminus \mathcal{D}$.

The results of this subsection will be completed by stating a result comparing the sum $\sum_{k=1}^\infty |\widehat{f}(k)|^p k^{p-1} \omega_k$ and the Dirichlet norm $\|f'\|_{A_{\widehat{\omega}_{[p-2]}}^p}^p$ under a natural hypothesis on ω in our setting. While not used in any of the proofs of the main results to be presented, it provides useful norm inequalities on its own. The proof relies on the Hardy-Littlewood inequalities, see [13, Theorems 6.2 and 6.3].

Proposition 4.3. *[54, Proposition 19] Let $0 < p < \infty$ and let ω be a radial weight such that $\widehat{\omega}_{[p-2]} \in \widehat{\mathcal{D}}$. Then the following statements hold:*

(i) If $0 < p \leq 2$, then $\sum_{k=1}^\infty |\widehat{f}(k)|^p k^{p-1} \omega_k \lesssim \|f'\|_{A_{\widehat{\omega}_{[p-2]}}^p}^p$ for all $f \in \mathcal{H}(\mathbb{D})$;

(ii) If $2 \leq p < \infty$, then $\|f'\|_{A_{\widehat{\omega}_{[p-2]}}^p}^p \lesssim \sum_{k=1}^\infty |\widehat{f}(k)|^p k^{p-1} \omega_k$ for all $f \in \mathcal{H}(\mathbb{D})$.

Multiple of the lemmas presented in this section state that the estimates therein are generally not valid for all functions in the class $\mathcal{S}$ if $\omega \in \widehat{\mathcal{D}} \setminus \mathcal{D}$. To see the reasoning behind this, consider the function $f(z) = -\log(1-z)$ along with the radial weight v_α defined in (3.3). Clearly $f \in \mathcal{S}$ and $v_\alpha \in \widehat{\mathcal{D}} \setminus \mathcal{D}$. With these choices, the integral quantities that define their respective function spaces and classes studied in this paper take the form

$$\|f\|_{X^p_{v_\alpha}} \asymp \int_0^1 \frac{dr}{(1-r)\left(\log \frac{e}{1-r}\right)^{\alpha+x}},$$

where the exponent x is dependent on the function space or class X. Straightforward calculations show that $x = 0$ for $A^p_{v_\alpha [p-1]}$ for the derivative $f'(z) = (1-z)^{-1}$ in view of Proposition 4.2. Similarly for f, we have $x = -p$ for $J^p_{v_\alpha}$, $x = -p/2$ for $S^p_{v_\alpha}$, and $x = 1-p$ for $H^p_{v_\alpha}$. The last asymptotic for $H^p_{v_\alpha}$ is a consequence of the fact that, for each $\delta > 0$, the function $t \mapsto t^\delta \log(2e^{\frac{1}{\delta}} t^{-1})$ is increasing on $(0, 2)$. For the sum appearing in Lemma 4.9, we have

$$\sum_{k=1}^{\infty} |\widehat{f}(k)|^p k^{p-1} (v_\alpha)_{2k} \asymp \sum_{k=1}^{\infty} k^{-1} \widehat{v_\alpha}\left(1 - \frac{1}{k}\right) \asymp \sum_{k=1}^{\infty} \frac{1}{k\left(\log(k+1)\right)^{\alpha-1}},$$

and similarly for the sum appearing in Lemma 4.10. Hence the parts of our results claiming that the presented estimates do not generally hold for all $f \in \mathcal{S}$ and $\omega \in \widehat{\mathcal{D}} \setminus \mathcal{D}$ can be shown by choosing α suitably depending on the comparison under inspection.

4.3.3 Main results

We turn to present the main results of the paper. The first few results concentrate on establishing asymptotic inequalities between weighted Bergman spaces in terms of what assumptions are necessary to demand from the respective radial weights. Note that multiple results are not restricted to conformal maps, but rather are valid for all analytic functions of the unit disc.

Theorem 4.8. *[54, Theorem 1] Let ω be a radial weight and $0 < p, q < \infty$. Then there exists a constant $C = C(p, q, \omega) > 0$ such that*

$$\|f\|_{A^p_{\omega_{[q-1]}}} \leq C \|f\|_{A^p_{\widehat{\omega}_{[q-2]}}}, \quad f \in \mathcal{H}(\mathbb{D}), \tag{4.10}$$

if and only if either $\widehat{\omega}_{[q-2]} \in \widehat{\mathcal{D}}$ or $\widehat{\omega}_{[q-2]} \notin L^1$.

The case $q = 1$ is valid by [49, Theorem 8], so our contribution consists of treating the other positive values of q. Moreover, for each $1 < q < \infty$, we have $\widehat{\omega}_{[q-2]} \in \widehat{\mathcal{D}}$ if and only if $\omega \in \widehat{\mathcal{D}}$ by Lemma 4.1(v). This equivalence is certainly false in general for $0 < q \leq 1$, because, for each $1 < \alpha < \infty$, the weight v_α defined in (3.3) belongs to $\widehat{\mathcal{D}}$, but $\widehat{v_\alpha}_{[-1]}$ is not even a weight if $\alpha \leq 2$. However, if $\omega \in \widehat{\mathcal{D}}$ and $\widehat{\omega}_{[q-2]} \in L^1$, then $\widehat{\omega}_{[q-2]} \in \widehat{\mathcal{D}}$ for each $0 < q < \infty$ by the proof of Lemma 4.1(v). The converse implication is also false because there exists a weight $\omega \notin \widehat{\mathcal{D}}$ such that $\widehat{\omega}_{[-1]} \in \widehat{\mathcal{D}}$ by [49, Theorem 3].

The estimate in Theorem 4.8 would not be much of use unless we were able to say when the two norms or seminorms are actually comparable. The next theorem establishes the norm comparability we are after.

Theorem 4.9. *[54, Theorem 2] Let ω be a radial weight and $0 < p, q < \infty$ such that $\widehat{\omega}_{[q-2]} \in L^1$. Then the following assertions hold:*

(i) If $0 < q < 1$, then

$$\|f\|_{A^p_{\widehat{\omega}_{[q-2]}}} \asymp \|f\|_{A^p_{\omega_{[q-1]}}}, \quad f \in \mathcal{H}(\mathbb{D}), \tag{4.11}$$

if and only if $\widehat{\omega}_{[q-2]} \in \widehat{\mathcal{D}}$.

(ii) If $q = 1$, then (4.11) is satisfied if and only if $\widehat{\omega}_{[-1]} \in \mathcal{D}$.

(iii) If $1 < q < \infty$, then (4.11) is satisfied if and only if $\omega \in \mathcal{D}$.

The next theorem concerns the converse of (4.10), to which we give a partial result. Proposition 4.1(ii) shows that, for each radial weight ω, $1 \leq q < \infty$ and $0 < p < \infty$, a necessary condition for the estimate

$$\|f\|_{A^p_{\widehat{\omega}_{[q-2]}}} \lesssim \|f\|_{A^p_{\omega_{[q-1]}}}, \quad f \in \mathcal{H}(\mathbb{D}), \tag{4.12}$$

to hold is $\omega \in \mathcal{M}$. We have been unable to judge if $\omega \in \mathcal{M}$ is also a sufficient condition for (4.12) to hold unless $p = 2$ (or $p = 2n$ for some $n \in \mathbb{N}$).

Theorem 4.10. *[54, Theorem 3] Let ω be a radial weight and $1 \leq q < \infty$. Then the following assertions are equivalent:*

(i) There exists a constant $C = C(\omega, q) > 0$ such that

$$\|f\|_{A^2_{\widehat{\omega}_{[q-2]}}} \leq C\|f\|_{A^2_{\omega_{[q-1]}}}, \quad f \in \mathcal{H}(\mathbb{D});$$

(ii) There exists a constant $C = C(\omega, q) > 0$ such that

$$\left(\widehat{\omega}_{[q-2]}\right)_x \leq C\left(\omega_{[q-1]}\right)_x, \quad 1 \leq x < \infty;$$

(iii) $\omega \in \mathcal{M}$.

The next result establishes a generalization to the well-known embeddings between D^p_{p-1} and H^p discussed in Section 3.2.1 as well as giving a concrete characterization of the weights the inequalities hold for. In addition, we show that the estimates require certain conditions from the weight ω with the doubling class $\widehat{\mathcal{D}}$ in the case of all analytic functions, but for conformal maps these assumptions are not necessary.

Theorem 4.11. *[54, Theorem 4] Let $0 < p < \infty$ and let ω be a radial weight. Then the following assertions hold:*

(i) If $0 < p < 2$, then $\|f\|_{H^p_\omega} \lesssim \|f'\|_{A^p_{\widehat{\omega}_{[p-2]}}}$ for all $f \in \mathcal{H}(\mathbb{D})$ if and only if $\widehat{\omega}_{[p-2]} \in \widehat{\mathcal{D}}$ or $\widehat{\omega}_{[p-2]} \notin L^1$;

(ii) If $2 < p < \infty$, then $\|f'\|_{A^p_{\widehat{\omega}_{[p-2]}}} \lesssim \|f\|_{H^p_\omega}$ for all $f \in \mathcal{H}(\mathbb{D})$ if and only if $\omega \in \widehat{\mathcal{D}}$.

Moreover, both norm estimates are valid for all $f \in \mathcal{S}$ without any hypotheses on ω.

In proving the last assertions for the class $\mathcal{S}$, we appeal to the growth and distortion theorems for functions in $\mathcal{S}$. Namely, by [14, Theorem 2.7],

$$\left|\frac{f'(z)}{f(z)}\right| \le \frac{1}{|z|}\frac{1+|z|}{1-|z|}, \quad z \in \mathbb{D}\setminus\{0\}, \quad f \in \mathcal{S},$$

and hence, for $2 \le p < \infty$ and $0 < r < 1$, we have

$$M_p^p(r, f') = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^{p-2}|f'(re^{i\theta})|^2 \left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right|^{p-2} d\theta \lesssim \frac{M_1(r, \Delta|f|^p)}{r^{p-2}(1-r)^{p-2}}.$$

The assertion for $2 \le p < \infty$ follows from this estimate and standard arguments. In a similar way we obtain

$$M_p^p(r, f') \gtrsim M_1(r, \Delta|f|^p) r^{2-p}(1-r)^{2-p}, \quad 0 < r < 1,$$

provided $0 < p < 2$. This yields the assertion concerning the class $\mathcal{S}$.

All the theorems and the estimates within so far have primarily been valid for all analytic functions of the unit disc. The remaining main results will turn our focus more towards conformal maps. We continue to present a result generalizing known results for classically weighted spaces in [22] and [52] to more general radial weights. We find that $\omega \in \mathcal{D}$ suffices to guarantee that the norms of univalent functions in H^p_ω and S^p_ω along with the maximum modulus integral quantity J^p_ω are comparable for all $0 < p < \infty$.

Theorem 4.12. *[54, Theorem 5] Let $0 < p < \infty$ and $\omega \in \mathcal{D}$. Then*

$$\|f\|^p_{H^p_\omega} + |f(0)|^p \asymp \|f\|^p_{S^p_\omega} + |f(0)|^p \asymp J^p_\omega(f), \quad f \in \mathcal{U}.$$

The following theorem provides yet another comparability result for univalent functions, now tying two differently weighted Dirichlet spaces with the quantities J^p_ω and $I_{p,q,\omega}$.

Theorem 4.13. *[54, Theorem 6] Let $2 \le p, q < \infty$ and $\omega \in \mathcal{D}$. Then*

$$\begin{aligned}\|f\|^p_{D^p_{\omega_{[p-1]}}} + |f(0)|^p \asymp \|f\|^p_{D^p_{\widehat{\omega}_{[p-2]}}} + |f(0)|^p \asymp J^p_\omega(f)& \\ \asymp I_{p,q,\omega}(f) + |f(0)|^p, \quad f \in \mathcal{U}.&\end{aligned}$$

We conclude this section with a result showing a comparability between the maximum modulus quantity J^p_ω and a sum of the Hardy-Littlewood type for univalent functions. Note that, for $\omega \in \widehat{\mathcal{D}}$, the sum appearing in the definition of the weighted Hardy-Littlewood space in (3.5) is comparable to the right hand side of (4.13), since an application of Lemma 4.1(iv) shows that in this case, for each fixed $0 < p < \infty$, the moments satisfy $\omega_{kp+1} \asymp \omega_k$ for all $k \in \mathbb{N}$.

Theorem 4.14. *[54, Theorem 7] Let $1 \le p \le 2$ and $\omega \in \mathcal{D}$. Then*

$$J^p_\omega(f) \asymp \sum_{k=0}^{\infty} |\widehat{f}(k)|^p (k+1)^{p-1}\omega_k, \quad f \in \mathcal{U}. \tag{4.13}$$

Moreover, this estimate is valid for all close-to-convex functions if $1 \le p < \infty$.

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TONI VESIKKO

This thesis contains several new results on conformal maps of the complex unit disc. The famous Becker's univalence criterion is considered with a linear error in the context of the Chuaqui-Stowe question, yielding various extensions to known classical results. Univalent functions of certain function spaces are considered, leading to improvements on characterizations related to Hardy, Bergman and Dirichlet spaces in addition to some geometrically defined spaces. Various norm inequalities are established for the aforementioned spaces together with applications to conformal maps.

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