ECONOPHYSICS Section

A DYNAMIC THEORY OF ECONOMICS: WHAT ARE THE MARKET FORCES?

Alia Asha DANNENBERG*, Matti ESTOLA** and Anna DANNENBERG***

Abstract. *The main weakness in the neoclassical theory of economics is its static nature. By a static model one cannot explain observed time paths of economic quantities, like the flows of production of firms, the flows of consumption of consumers, and the prices of goods. The error in the neoclassical framework is that there economic units are assumed to be in their optimum state and thus not willing to change their behavior. Therefore, in neoclassical models a static equilibrium prevails. In this paper, the authors change this assumption so that economic units are assumed to be willing to improve their current state that may not be the optimal one. In this way, one can explain economic dynamics where every economic unit is changing its behavior towards improving its welfare. The authors define the economic forces acting upon the production of firms, the consumption of consumers, and the prices of goods. They show that in this dynamic system, business cycles and bankruptcies of firms emerge in a natural way like in the real world.*

Keywords: *Econophysics; Newtonian economics; dynamic economics.*

JEL: *D11, D21, C63, C02.*

1. Introduction

The fundamental weakness in current macroeconomic theory is the absence of a consistent micro level foundation. Here we present a new microeconomic theory where the macro state of a system is the aggregate of states of the micro units as proposed by Lux and Westerhoff (2009) in the spirit of classical analytical mechanics. Throughout our framework, we define and apply a consistent unit system for economics presented by De Jong (1967), comparable to that of physics.

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* Lukema Oy, Joensuu, Finland, alia.dannenberg@lukema.eu

** University of Eastern Finland, matti.estola@uef.fi

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As e.g. Lux and Westerhoff (2009) state, the neoclassical economic theory is widely known of its inability to model the behavior of real economic phenomena. The most fundamental shortcoming in the prevailing neo-classical framework, acknowledged e.g. by Mas-Colell et al. (1995), is that it is essentially static in nature, whereas real economic systems are always dynamic. There have been attempts to dynamize the neoclassical theory for consumers by e.g. Ramsey (1928); Cass (1965); Koopmans (1965) and for firms by Evans (1924), but these theories are, according to e.g. Estola (2013), inconsistent with the static neo-classical framework. Within the neoclassical framework, economic units are assumed to be in their optimum state, resulting in that the equations do not cover situations outside the optimum.

This article introduces a dynamic theory of economics, compatible with real economic phenomena. It can be considered as a dynamic extension to the neoclassical framework, including the latter one as a special case with a static setup. Our theory can explain observed dynamic economic phenomena also outside optimum states. Therefore, it can be used to simulate economic systems in a realistic way such as economic crises that the neo-classical framework is unable to forecast or handle, see Lux and Westerhoff (2009). Our theory has been tested with extensive simulations and two empirical evaluations, and it has been found consistent with real data, as shown in Estola and Dannenberg (2012); Estola (2015).

2. Firm and production

In building our theory, let us begin from the basics. Let the profit Π with unit €/*time* of a multi-product firm under perfect competition be

$$\Pi = \sum_{k=1}^{K} P_k \dot{Q}_k - C(\dot{\boldsymbol{Q}}), \tag{1}$$

where $\dot{Q}_k$ is the flow of production of good k with unit $piece_k/time$, P_k the price of product k with unit $€/piece_k$, and $C(\dot{\boldsymbol{Q}})$ with unit $€/time$ the costs of the firm at the production flow vector $\dot{\boldsymbol{Q}} = (\dot{Q}_1, \dots, \dot{Q}_K)$. Now, the optimum conditions are

$$\frac{\partial \Pi}{\partial \dot{Q}_k} = P_k - \frac{\partial C(\dot{\boldsymbol{Q}})}{\partial \dot{Q}_k} = 0, \quad k = 1, \dots, K, \tag{2}$$

where $\partial C(\dot{\boldsymbol{Q}})/\partial \dot{Q}_k$ with unit $€/piece_k$ denotes marginal costs.

This is the neoclassical optimum that corresponds to the “zero-force” situation in Newtonian mechanics (Newton’s first law): “A body does not change its state of motion unless there is a force acting upon it”. If the flows of production of a firm yield the maximum profit, the firm does not want to change them. But what if $P_k - \partial C(\dot{\boldsymbol{Q}})/\partial \dot{Q}_k \neq 0$? Then the firm is not in its optimum and, according to Fisher (1983); Varian (2006), the firm should either increase or decrease its flow of production of good k to gain higher profit. For a profit-seeking firm, there is then a "force" that drives it to adjust its flow of production. In the neoclassical theory, these economic forces have been acknowledged but never defined exactly which, according to Mirowski (1989), has led to the static framework.

Definition 1: Quantity $F_{\dot{Q}_k} = P_k - \partial C(\dot{\boldsymbol{Q}})/\partial \dot{Q}_k$ is the economic force acting upon the production of good k of the firm. The unit of this economic force is the same as the unit of price, i.e., $€/piece_k$.

Definition 2: $F_{\dot{Q}_k} = m_{\dot{Q}_k}\ddot{Q}_k$. Economic force $F_{\dot{Q}_k}$ causes either positive or negative acceleration on production of good k; if $F_{\dot{Q}_k} > 0$ then $\ddot{Q}_k > 0$, and vice versa.

This is similar with Newton's second law. Q_k is the accumulated amount of production of good k, $\ddot{Q}_k$ the acceleration of accumulated production, and $m_{\dot{Q}_k}$ the inertia of production[1] (it takes time to speed up or wind down production). The unit of this positive inertia term is $€ \times time^2/piece_k^2$.

The work done by the economic force acting upon production can be calculated like the work of physical force, see Estola and Dannenberg (2016):

$$\begin{aligned}\Delta W &= \int F_{\dot{Q}} \cdot d\boldsymbol{Q} \\ &= \sum_k m_{\dot{Q}_k} \int \ddot{Q}_k dQ_k \\ &= \sum_k m_{\dot{Q}_k} \int \frac{d\dot{Q}_k}{dt} dQ_k = \sum_k m_{\dot{Q}_k} \int \frac{dQ_k}{dt} d\dot{Q}_k \\ &= \sum_k \frac{1}{2} m_{\dot{Q}_k} \dot{Q}_{k,final}^2 \\ &\quad - \sum_k \frac{1}{2} m_{\dot{Q}_k} \dot{Q}_{k,initial}^2. \end{aligned} \tag{3}$$

[1] We are assuming that inertias of production and consumption are time-independent. Time-dependent masses are possible, though would complicate the equations.

Eq. (3) suggests that kinetic energy of production exists in the production process. The unit of economic kinetic energy and work is €, see e.g., Dragulescu and Yakovenko (2000); Kusmartsev (2011). On the other hand, work ΔW_k is

$$\Delta W_k = \int_{Q_{k,i}}^{Q_{k,f}} F_{\dot{Q}_k} dQ_k$$
$$= \int_{t_i}^{t_f} \left(P_k \dot{Q}_k - \dot{Q}_k \frac{\partial C}{\partial \dot{Q}_k} \right) dt = \int_{t_i}^{t_f} \dot{Q}_k \left(P_k - \frac{\partial C}{\partial \dot{Q}_k} \right) dt,$$

where $\int_{t_i}^{t_f} P_k \dot{Q}_k dt$ is the revenues from sales of good k and the term $\int_{t_i}^{t_f} \dot{Q}_k \, \partial C/\partial \dot{Q}_k \, dt$ represents the costs within the time interval $\Delta t = t_f - t_i$ if unit cost equals marginal cost. If$P_k > \partial C/\partial \dot{Q}_k$, force $F_{\dot{Q}_k}$ does work to change the kinetic state of production, increasing the flow of production (note that $\dot{Q}_k \geq 0$). However, if $P_k < \partial C/\partial \dot{Q}_k$ production does work against the force$F_{\dot{Q}_k}$.

3. Consumer and consumption

For a consumer, the corresponding definitions are the following:

Definition 3: There exists a force acting upon the consumption of a consumer of good *k*: $F_{\dot{X}_k} = \partial H(\dot{\boldsymbol{X}})/\partial \dot{X}_k - P_k$.

This is similar with Definition 1. $H(\dot{\boldsymbol{X}})$ with unit $€/time$ is the willingness to pay of a consumer for the consumption flow vector $\dot{\boldsymbol{X}}$, and $\partial H(\dot{\boldsymbol{X}})/\partial \dot{X}_k$ the marginal willingness to pay of the consumer that corresponds to the marginal costs of a firm, see Dannenberg and Estola (2018). The consumer surplus (measured in unit$€/time$ like the profit of a firm) is $\phi = H(\dot{\boldsymbol{X}}) - \sum_{k=1}^{K} P_k \dot{X}_k$, and its optimum corresponds to the zero force acting upon consumption, i.e., $F_{\dot{X}_k} = 0 \Leftrightarrow \partial H(\dot{\boldsymbol{X}})/\partial \dot{X}_k = P_k$. Consumers' marginal willingness to pay for a good can be measured, e.g., by making a consumer survey, see Cameron and James (1987).

Definition 4: The force acting upon consumption causes either positive or negative acceleration in consumption, i.e., $F_{\dot{X}_k} = m_{\dot{X}_k} \ddot{X}_k$.

This is similar with Definition 2. $\ddot{X}_k$ is the acceleration of consumption of good k andthe consumption of good k has kinetic energy $\frac{1}{2}m_{\dot{X}_k}\dot{X}_k^2$. Since according to e.g. Dannenberg and Estola (2018) the theories of a firm and a consumer are symmetric and have the same mathematical form, similar work effects for force $F_{\dot{X}_k}$ are obtained for consumer dynamics.

4. Market mechanism as a spring system

Let us consider a simple physical problem: two masses, say m_X and m_Q, are attached to each other with a spring with spring constant k, length L and rest length zero.

Figure 1. A simple spring system.

The masses have initial velocities $\dot{X}_0$ and $\dot{Q}_0$ towards the direction of the symmetry axis. Moreover, m_X is drawn by force F_X and there is dragging force $-F_Q$ affecting m_Q. Figure 1 illustrates the setup. The forces are, according to Hooke's law,

$$m_Q\ddot{Q} = F_k - F_Q, \tag{5}$$

$$m_X\ddot{X} = F_X - F_k, \tag{6}$$

$$F_k = kL = k(X - Q). \tag{7}$$

Now one can calculate the time evolution of the system.

As Definitions 1 and 3 state, the forms of the economic forces are similar with the spring forces. Most notably, price P_k resembles the harmonic force F_k that in economics connects production with consumption. We observe the following similarities: (i) $F_k \sim P_k$. (ii) $F_Q \sim \partial C(\dot{\boldsymbol{Q}})/\partial\dot{Q}_k$, and $F_X \sim \partial H(\dot{\boldsymbol{X}})/\partial\dot{X}_k$. Marginal costs and marginal willingness to pay are external forces. (iii) The time derivative of Eq. (7) yields the law of demand and supply $\dot{P}_k = k(\dot{X}_k - \dot{Q}_k)$ that relates price changes to excess demand or supply as shown e.g. by Samuelson (1941, 1942). Forces $\partial C(\dot{\boldsymbol{Q}})/\partial\dot{Q}_k$ and $\partial H(\dot{\boldsymbol{X}})/\partial\dot{X}_k$ depend on the corresponding velocities. (iv) Natural constraints are $\dot{X}_k \geq 0$, $\dot{Q}_k \geq 0$, and $P_k \geq 0$ because it is impossible to produce or consume negative amounts of goods or pay negative prices.

Definition 5: Price P_k is a harmonic force that connects the flows of production and consumption.

Recall Hooke's law and Definitions 1 and 3. Price P_k is an external force for individual consumers and firms because they all participate on determining the "right" price.

Definition 6: For each action, there is equal reaction in opposite direction.

The law of mutual forces of action and reaction (Newton's third law) holds in economics as well. The sum of forces of a closed system is zero. However, most economic systems are open, as is our simulation example. The whole real global economy is naturally a closed system.

5. Simulated economic crises

Now, one can construct an arbitrarily large system consisting of $i = 1, \dots, I$ firms, $j = 1, \dots, J$ consumers and $k = 1, \dots, K$ goods. The equations governing the dynamics are

$$m_{\dot{Q}_{i,k}} \ddot{Q}_{i,k} = P_k - \frac{\partial C_i(\dot{\boldsymbol{Q}})}{\partial \dot{Q}_{i,k}}, \tag{8}$$

$$m_{\dot{X}_{j,k}} \ddot{X}_{j,k} = \frac{\partial H_j(\dot{\boldsymbol{X}})}{\partial \dot{X}_{j,k}} - P_k, \tag{9}$$

$$\frac{1}{k_{P_k}} \dot{P}_k = \sum\nolimits_{j=1}^{J} \dot{X}_{j,k} - \sum\nolimits_{i=1}^{I} \dot{Q}_{i,k}. \tag{10}$$

The law of demand and supply in Eq. (10) is familiar from the neo-classical theory, see e.g. Samuelson (1941, 1942). Equations governing production (8) and consumption (9) do not exist in the neo-classical theory, but they are fundamental in the dynamic theory of economics, see Estola and Hokkanen (2008); Estola (2017). The effects of production and consumption on prices cannot be treated separately but by using the whole system. The neoclassical optimum is obtained by setting all masses $m_{\dot{X} \wedge \dot{Q}} \to 0$ and $1/k_{P_k} \to 0$.

For simulating economic crises, we use a standard cost function $C_i(\dot{\boldsymbol{Q}}) = A_i + \sum_k \left(B_{i,k} \dot{Q}_{i,k} + D_{i,k} \dot{Q}_{i,k}^2\right)$ for firm i, where A_i represents fixed costs and $B_{i,k}$, $D_{i,k}$ are the constants of variable costs. Each firm produces three randomly chosen products. Moreover, we use utility function for consumer j

$$U_j = u_j\left(\left(1 - exp\left(-E_{j,1}\,\dot{X}_{j,1}/\dot{X}_{j,10}\right)\right) + \left(1 - exp\left(-E_{j,2}\,\dot{X}_{j,2}/\dot{X}_{j,20}\right)\right) + \cdots + \left(1 - exp\left(-E_{j,K}\,\dot{X}_{j,K}/\dot{X}_{j,K0}\right)\right)\right) = u_j\sum_{k=1}^{K}\left(1 - exp\left(-E_{j,k}\,\dot{X}_{j,k}/\dot{X}_{j,k0}\right)\right)$$

that obeys positive and decreasing marginal utility for all goods $k = 1, \ldots, K$,

$$\frac{\partial U_j}{\partial \dot{X}_{j,k}} = u_j\frac{E_{j,k}}{\dot{X}_{j,k0}}e^{-E_{j,k}\frac{\dot{X}_{j,k}}{\dot{X}^0_{j,k}}} > 0, \qquad \frac{\partial^2 U_j}{\partial \dot{X}^2_{j,k}} = -u_j\frac{E^2_{j,k}}{\dot{X}^2_{j,k0}}e^{-E_{j,k}\frac{\dot{X}_{j,k}}{\dot{X}^0_{j,k}}} < 0.$$

Constant $E_{j,k}$ measureshow much consumer j prefers good k and $\dot{X}_{j,k0}$ is the initial consumption of good k of consumer j. Our utility function has some advantages over the usual logarithmic utility functions used by e.g. Varian (2006), most notably that utility can never be infinite. $U_j = 0$ if $\dot{X}_{j,k} = 0\ \forall\ k$ and the maximum of U_j is Ku_j if $\dot{X}_{j,k} = \infty\ \forall\ k$. This function is realistic for people's real consumption.

The willingness to pay function of consumer j for his/her consumption flow vector $\dot{\boldsymbol{X}}_j$ is $H_j(\dot{\boldsymbol{X}}_j) = \eta_j(M_j)U_j(\dot{\boldsymbol{X}}_j)$, where η_j with unit $€/util$ is the utility-money conversion factor of consumer j,

$$\eta_j = F_j\left(1 - exp\left(-\,G_j M_j/< M >\right)\right).$$

Factor η_j obeys the law of decreasing marginal utility of money assumed e.g. by Bernoulli (1738); von Neumann and Morgenstern (1953). The bigger is η_j, the more consumer j is willing to pay for his consumption flow vector $\dot{\boldsymbol{X}}_j$, that is, the more he/she values consumption over saving. The bigger are positive constants F_j, G_j and the higher is the income ratio of consumer j to average income, $M_j/< M >$, the more consumer j prefers spending over saving. $M_j = M_{j0} + \rho_j r W_j$ is the income of consumer j that consists of labor income M_{j0} and interest earnings or payments on wealth rW_j depending on whether $W_j > 0$ or $W_j < 0$. The maximum of η_j is F_j if $M_j = \infty$, and the minimum of η_j is 0 if $M_j = 0$. We assume enough consumers that the effect of the income of consumer j on the average income can be neglected. Then we get

$$\frac{\partial \eta_j}{\partial M_j} = F_j\frac{G_j}{< M >}e^{-G_j\frac{M_j}{<M>}} > 0, \qquad \frac{\partial^2 \eta_j}{\partial M_j^2} = -F_j\frac{G_j^2}{(< M >)^2}e^{-G_j\frac{M_j}{<M>}} < 0.$$

Thus, with higher income consumer j is willing to pay more for his/her consumption flow vector $\dot{\boldsymbol{X}}_j$, but raising income increases factor η_j in a decreasing way (see Dannenberg and Estola (2018) for details). For simplicity, the same rate is assumed for interest revenues and costs and ρ_j is a consumer specific factor that magnifies or dilutes the wealth effect if $\rho_j > 1$ or $\rho_j < 1$.[2]

The consumer spending problem is an open optimization problem with soft boundaries like the optimization problem of a firm, see Dannenberg and Estola (2018). Consumers may use credit or spend only a part of their income, depending on their marginal willingness to pay and the prices of goods. The marginal willingness to pay $\partial H_j(\dot{\boldsymbol{X}}_j)/\partial \dot{X}_{j,k}$ is calculated from the willingness to pay function $H_j(\dot{\boldsymbol{X}}_j) = \eta_j(M_j)U_j(\dot{\boldsymbol{X}}_j)$. "Masses" $m_{\dot{X}_{j,k}}$, $m_{\dot{Q}_{i,k}}$ are the inertias of consumption and production, and k_{P_k} is the spring constant of price P_k.

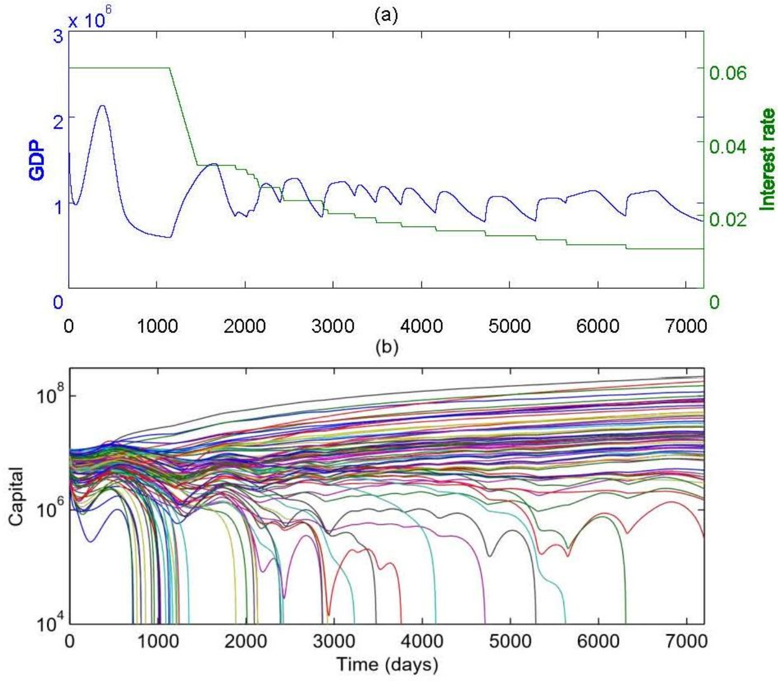

Figure 2. (a) GDP and central bank interest rate of the simulated economy. Lowering interest rate temporarily increases GDP. (b) Capitals of firms. If the capital of a firm decreases to zero, the firm is declared into bankruptcy. The amount of bankruptcies affects the interest rate adjusted by central bank.

[2] The units of our parameters and new functions are: $[U_j \wedge u_j]$: $util/time$; $[\eta_j \wedge F_j]$: $€/util$; $[M_j \wedge M_{j0}]$: $€/time$; $[W_j]$: €; $[r]$: $1/time$, and $E_{j,k}$, G_j, ρ_j are pure numbers.

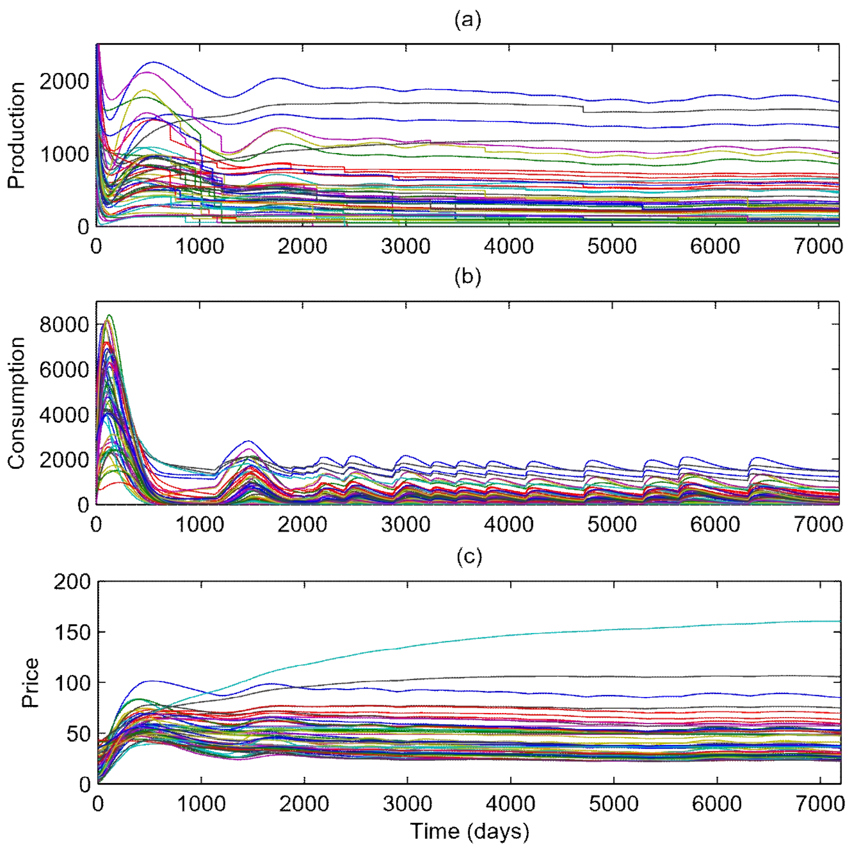

Figure 3. (a) Production, (b) consumption and (c) price of goods in the simulated economy. A business cycle of 300-500 days is clearly visible in consumption. Production and prices follow consumption, but they are slower to react to changes in the market.

With fixed initial conditions $\dot{X}_{j,k0}$,$\dot{Q}_{i,k0}$ and $P_{k0} \forall\ i, j, k,$ one can solve the dynamics of the system by numerical simulations. For simplicity, we set the initial wealth of consumer j to zero, $W_{j0} = 0$, and the firms have randomly distributed initial capitals. The system is open because income and interest earnings come from outside the system, and costs and interest payments go outside the system; it is possible, though, to make the system closed. Unlike many economic models, bankruptcies are allowed: in this our first approximation model, a firm that loses all of its capital is declared into bankruptcy and taken instantly out of the market by setting its production to zero. Moreover, there is an exogenous "central bank" that sets the interest rate r by economic conditions, basically based on the bankruptcy rate of firms.

In our simulations, we have created an economy of 100 firms, 50 goods and 5000 consumers and simulated it for 7200 days. All parameters in the model are balloted from a uniformly distributed interval before every simulation, and they are not changed during the simulation. The masses of production are assumed roughly ten times bigger than those of consumption. The model *GDP* is calculated as $GDP = \sum_i \left(\sum_k P_k \dot{Q}_{i,k} \right)$.

Figures 2 and 3 illustrate the results. The most notable result is that from the complexity of many interacting agents, business cycles emerge naturally. This can be understood by the physical model in Figure 1 because it is well-known that the adjustment in a spring system finds its equilibrium via oscillations. The duration of a business cycle depends on the initial conditions; with our initial values, it varies roughly between 300 and 500 days. The business cycles are most clearly visible in the figures representing *GDP* (2a) and consumption (3b) of the economy. Notice that we have assumed no technical progress in production which would propel economic growth in time, see e.g. Estola (2001).

The central bank actions suppress natural business cycles in production and prices. Figure 2 shows that each time the interest rate is lowered, GDP temporarily increases, but it quickly returns to its original level or even lower demanding a new intervention. While production and prices have greater inertia than consumption, the central bank interventions prevent them from getting low enough for the next healthy expansion phase of the economy. Instead, they are supported by lower interest rate that allows consumers more credit. The interest rate policy is effective in saving firms from bankruptcy (see Fig. 2), but it results in over-indebtedness of consumers.

The whole economy is thus highly dependent on central bank actions. In the end of our simulation, the interest rate is lowered to ≈1%/year, and there is not much ammo left in the central bank's interest rate arsenal. (The real global economy passed this mark a couple of years ago, causing central banks to adopt extraordinary measures such as "quantitative easing".) In our model, central bank actions seem to improve the short-term economic situation but ultimately result in economic crises.

Our title asks: "What are the market forces?". Our analysis gives a well-defined answer: the market forces are marginal costs, marginal willingness's to pay and prices.

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