2020

On the periodicity of transcendental entire functions

Liu, Xinling

Cambridge University Press (CUP)

Tieteelliset aikakauslehtiartikkelit

http://dx.doi.org/10.1017/S0004972720000039

Bull. Aust. Math. Soc. (First published online 2020), page 1 of 13*
doi:10.1017/S0004972720000039
*Provisional—final page numbers to be inserted when paper edition is published

ON THE PERIODICITY OF TRANSCENDENTAL ENTIRE FUNCTIONS

XINLING LIU and RISTO KORHONEN✉

(Received 11 June 2019; accepted 27 November 2019)

Abstract

According to a conjecture by Yang, if $f(z)f^{(k)}(z)$ is a periodic function, where $f(z)$ is a transcendental entire function and k is a positive integer, then $f(z)$ is also a periodic function. We propose related questions, which can be viewed as difference or differential-difference versions of Yang's conjecture. We consider the periodicity of a transcendental entire function $f(z)$ when differential, difference or differential-difference polynomials in $f(z)$ are periodic. For instance, we show that if $f(z)^n f(z+\eta)$ is a periodic function with period c, then $f(z)$ is also a periodic function with period $(n+1)c$, where $f(z)$ is a transcendental entire function of hyper-order $\rho_2(f) < 1$ and $n \geq 2$ is an integer.

2010 *Mathematics subject classification*: primary 30D35.
Keywords and phrases: entire functions, periodicity, growth.

1. Introduction and main results

Periodicity is an important and easy to recognise property for meromorphic functions. Rényi and Rényi [15] proved that if $f(z)$ is an arbitrary nonconstant entire function and $P(z)$ is an arbitrary polynomial with $\deg(P(z)) \geq 3$, then the entire function $f(P(z))$ cannot be a periodic function. If $\deg(P(z)) = 2$, then there exists a transcendental entire function $f(z)$ such that $f(P(z))$ is periodic. For example, if $P(z) = Az^2 + Bz + C$, where $A \neq 0$, B, C are constants and

$$f(z) = \cos\sqrt{4A(z-C)+B^2} = \sum_{k=0}^{\infty}(-1)^k \frac{(4A(z-C)+B^2)^k}{(2k)!},$$

then

$$f(P(z)) = \cos(2Az + B)$$

is a periodic function with period π/A. Rényi and Rényi [15] also proved that if $Q(z)$ is a nonconstant polynomial and $g(z)$ is entire and nonperiodic, then $Q(g(z))$ cannot be periodic either. Thus, if $Q(g(z))$ is a periodic function, then also $g(z)$ must be a periodic

The first author was partially supported by the NSFC (No. 11661052) and EDUFI Fellowship No. 18-11020. The second author was partially supported by the Academy of Finland Grant No. 286877.

function. Further investigations on the periodicity of entire functions can be found in [1, 5, 6, 18].

Ozawa [14, Theorem 1] has shown that for any $\rho \in [1, +\infty)$ there exists a prime periodic entire function h of order $\rho(h) = \rho$. We assume that the reader is familiar with the basic symbols and fundamental results of Nevanlinna theory [8, 19]. Recall that the order of $f(z)$ is defined by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r}$$

and the hyper-order of $f(z)$ is defined by

$$\rho_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r, f)}{\log r}.$$

Given a nonconstant meromorphic function f, the family of all meromorphic functions w such that $T(r, w) = o(T(r, f))$, where $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure, is denoted by $S(f)$. Let $\widehat{S}(f) = S(f) \cup \{\infty\}$. Suppose that f, g are meromorphic and $a \in \widehat{S}(f)$. Denoting by $E(a, f)$ the set of those points $z \in \mathbb{C}$ where $f(z) = a$, we say that f, g share a IM (ignoring multiplicities) if $E(a, f) = E(a, g)$. Provided that $E(a, f) = E(a, g)$ and the multiplicities of the zeros of $f(z) - a$ and $g(z) - a$ are the same at each $z \in \mathbb{C}$, then f, g share a CM (counting multiplicities).

Heittokangas *et al.* [9, Theorem 2] obtained the periodicity of $f(z)$ under the condition that $f(z)$ and $f(z + c)$ share three small periodic functions.

Theorem A. *Let $f(z)$ be a finite-order transcendental meromorphic function and let $a_1, a_2, a_3 \in \widehat{S}(f)$ be three distinct periodic functions with period c. If $f(z)$ shares a_1, a_2 CM and a_3 IM with $f(z + c)$, then $f(z) = f(z + c)$ for all $z \in \mathbb{C}$.*

We consider the periodicity of an entire function $f(z)$ when a differential, difference or differential-difference polynomial in $f(z)$ is periodic. We assume that n, k are integers in the following. Note that $f^{(k)}(z)$ $(k \geq 1)$ can be a periodic function even if $f(z)$ is not periodic. For instance, $f(z) = e^z + z$ is an example of such a function. However, replacing $f(z)$ by $f(z)^n$ $(n \geq 2)$ or $f(z)^n f(z + c)$ $(n \geq 2)$, the periodicity can be determined partially. The questions posed in the present paper are inspired by Yang's conjecture, which appeared firstly in [16, Conjecture 1.1].

Yang's conjecture. *Let $f(z)$ be a transcendental entire function and k be a positive integer. If $f(z)f^{(k)}(z)$ is a periodic function, then $f(z)$ is also a periodic function.*

Wang and Hu [16, Theorem 1.1] showed that Yang's conjecture is true for $k = 1$, while Liu and Yu [13, Theorem 1.1] proved that Yang's conjecture is also true for an arbitrary k if $f(z)$ has a nonzero Picard exceptional value, namely, if $f(z) = e^{h(z)} + d$, where $h(z)$ is a nonconstant entire function and d is a nonzero constant. Note that if $h(z)$ is a nonconstant polynomial and $d = 0$, Yang's conjecture is also true and this can be seen as follows. We assume that

$$f(z)f^{(k)}(z) = f(z + c)f^{(k)}(z + c),$$

where c is a nonzero constant. Substituting $f(z) = e^{h(z)}$ into the equation above gives $e^{2h(z+c)-2h(z)} = H(z)/H(z+c)$, where $H(z)$ is a polynomial in $h(z)$ and its derivatives and so also a polynomial in z. Since the rational function $H(z)/H(z+c)$ has no zeros and poles, then $H(z)/H(z+c) \equiv 1$. Thus, $e^{2h(z+c)-2h(z)} \equiv 1$, that is, $f(z)$ is a periodic function with period c or $2c$. Yang's conjecture for entire functions with a Picard exceptional value remains open in the case when $h(z)$ is transcendental and $d = 0$. We obtain the following result related to this question.

THEOREM 1.1. *Let $f(z) = p(z)e^{h(z)} + q(z)$, where $p(z), q(z)$ are nonzero polynomials and $h(z)$ is a nonconstant entire function. If $f(z)f^{(k)}(z)$ is a periodic function, then $p(z)$ and $q(z)$ are constants.*

Even though Yang's conjecture has not been completely solved, it inspires us to propose related questions which will be considered in the paper.

QUESTION 1.2. Let $f(z)$ be a transcendental entire function and n, k be integers. If $f(z)^n f^{(k)}(z+\eta)$ is a periodic function, does it follow that $f(z)$ is also a periodic function?

We begin to consider Question 1.2 in the case $\eta = 0$ when k is a positive integer (the case $\eta = 0$ and $k = 0$ is trivial). This is the differential version of Question 1.2 and a generalisation of Yang's conjecture. As we have seen, the case $n = 1$ and $k = 1$ has been solved by Wang and Hu [16, Theorem 1.1]. If $n \geq 2$ and $k = 1$, the answer to Question 1.2 is also positive. Namely, assuming that $f^n(z)f'(z)$ is a periodic function with period $c(\neq 0)$, then

$$f(z+c)^n f'(z+c) = f(z)^n f'(z),$$

which implies that

$$f(z+c)^{n+1} - f(z)^{n+1} = A, \tag{1.1}$$

where $A \in \mathbb{C}$. Equation (1.1) has no nonconstant entire solutions provided that $A \not\equiv 0$, which is a direct consequence of Yang's result [17, Theorem 1], that is, there are no nonconstant entire solutions $f(z)$ and $g(z)$ that satisfy $a(z)f(z)^n + b(z)g(z)^m = 1$ provided that $m^{-1} + n^{-1} < 1$, where $a(z), b(z) \in S(f)$. Hence, $A \equiv 0$ in (1.1) and $f(z+c) = tf(z)$, where $t^{n+1} = 1$. Thus, $f(z)$ is a periodic function with period $(n+1)c$. It remains open whether Question 1.2 is true for $k \geq 2$, $n \geq 2$.

We next consider the case $k = 0$ and $\eta \neq 0$ in Question 1.2, which is the difference version of Question 1.

THEOREM 1.3. *Let $f(z)$ be a transcendental entire function with $\rho_2(f) < 1$ and $n \geq 2$ be a positive integer. If $f(z)^n f(z+\eta)$ is a periodic function with period c, then $f(z)$ is a periodic function with period $(n+1)c$.*

Theorem 1.3 is not valid for transcendental entire functions with $\rho_2(f) \geq 1$. This can be seen by taking a nonperiodic entire function $f(z) = e^{ze^z}$ such that $e^\eta = -n$, where n is a positive integer. Then $f(z)^n f(z+\eta) = e^{-n\eta e^z}$ is a periodic function. We claim that

$f(z) = e^{ze^z}$ is not a periodic function. Otherwise, there exists a nonzero constant c such that $e^{ze^z} = e^{(z+c)e^{z+c}}$ and thus $(z+c)e^{z+c} - ze^z = 2k\pi i$, which is impossible for a nonzero constant c.

In the case $n = 1$, it is easy to see that if $f(z)f(z+\eta)$ is a periodic function with period $c_1 = \eta$, then $f(z)$ is also a periodic function with period 2η. However, the case $c_1 \neq \eta$ is still open.

We pose another question and obtain two results below.

QUESTION 1.4. Let $f(z)$ be a transcendental entire function and n, k be positive integers. If $[f(z)^n f(z+\eta)]^{(k)}$ is a periodic function, does it follow that $f(z)$ is also a periodic function?

THEOREM 1.5. *Let $f(z)$ be a transcendental entire function with $\rho_2(f) < 1$ and $n \geq 4$ be a positive integer. If $[f(z)^n f(z+\eta)]^{(k)}$ is a periodic function with period c, then $f(z)$ is a periodic function with period $(n+1)c$.*

Theorem 1.5 is not true if $n = 1$ and $k \geq 2$. This can be seen by the example $f(z) = e^z + z$ and $e^c = -1$, where $[f(z)f(z+c)]'' = -2^2e^{2z} + ce^z + 2$ and $[f(z)f(z+c)]^{(k)} = -2^k e^{2z} + ce^z$ $(k \geq 3)$ are both periodic functions with period $2c$, but $f(z)$ is not a periodic function. However, we have the following result.

THEOREM 1.6. *Suppose that $[f(z)^n f(z+\eta)]^{(k)}$ is a periodic function with period η. If $f(z)$ is a transcendental entire function of finite order and $n \geq 1$, then $f(z)$ is a periodic function with period $(n+1)\eta$. If $f(z)$ is a transcendental entire function of infinite order and $n = 1$, $k = 1$, then $f(z)$ is a periodic function with period 2η.*

Yang's conjecture and Question 1.2 are related to differential (difference or differential-difference) monomials and Question 1.4 is related to differential-difference polynomials. We will next consider the following Question 1.7 related to the derivatives of difference polynomials.

QUESTION 1.7. Let $f(z)$ be a transcendental entire function and $\Delta_\eta f := f(z+\eta) - f(z)$. If $[f(z)^n \Delta_\eta f]^{(k)}$ is a periodic function, does it follow that $f(z)$ is also a periodic function?

THEOREM 1.8. *Let $f(z)$ be a transcendental entire function with $\rho_2(f) < 1$ and $n \geq 5$ be a positive integer. If $[f(z)^n \Delta_\eta f]^{(k)}$ is a periodic function with period η, then $f(z)$ is a periodic function with period $(n+1)\eta$.*

Finally, observe that $f^{(k)}(z) + f^{(l)}(z)$ $(k > l)$ may be a periodic function, even if $f(z)$ is not a periodic function. This can be seen, for instance, by taking $f(z) = e^z + z$. On the relation between the periodicity of $f^{(k)}(z) + f^{(l)}(z)$ and $f(z)$, we give the following result.

THEOREM 1.9. *Let $f(z)$ be a transcendental entire function and let $(f(z)^2)^{(k)} + (f(z)^2)^{(l)}$ be a periodic function with period c. If $k = 1$ and $l = 0$, then $f(z)$ is a periodic function with period c, $2c$, $4c$ or $4i\pi$. If $\rho(f) \geq 2$ and $k > l$, then $f(z)$ is a periodic function with period $2c$ or $4c$.*

We see that Theorem 1.9 is not true for $\rho(f) = 1$, $k > l \geq 2$. Take $f(z) = e^{-z} + z + 1$. By an elementary computation, we see that $(f(z)^2)''' + (f(z)^2)'' = -4e^{-2z} + 2e^{-z} + 2$ is a periodic function, but $f(z)$ is not periodic. The case of $k = l$ is [16, Theorem 1.1].

2. Lemmas

The relations between the characteristic functions of a meromorphic function f and its difference polynomials will play important roles in our proofs. We firstly recall that if $f(z)$ is a transcendental entire function such that $\rho_2(f) < 1$, then

$$T(r, f(z+c)) = T(r, f) + S(r, f) \tag{2.1}$$

and

$$T(r, f(z+c) - f(z)) \leq T(r, f) + S(r, f). \tag{2.2}$$

These can be obtained by the difference analogue of the logarithmic derivative lemma [7, Lemma 8.3]. In the proofs of Theorem 1.5 and Theorem 1.8 below, the following three lemmas are needed.

Lemma 2.1 [12, Lemma 2.4]. *Let $f(z)$ be a transcendental entire function such that $\rho_2(f) < 1$. If $n \geq 1$, then*

$$T(r, f(z)^n f(z+c)) = (n+1)T(r, f) + S(r, f).$$

Lemma 2.2 [12, Lemma 2.6]. *Let $f(z)$ be a transcendental entire function such that $\rho_2(f) < 1$. If $n \geq 1$, then*

$$T(r, f(z)^n \Delta_\eta f) \geq nT(r, f) + S(r, f).$$

Lemma 2.3 [19, Theorem 1.62]. *Suppose that $n \geq 3$ and f_j $(j = 1, 2, \ldots, n)$ are meromorphic functions which are not constants except possibly for f_n. Let $\sum_{j=1}^{n} f_j = 1$. If $f_n \not\equiv 0$ and*

$$\sum_{j=1}^{n} N\Big(r, \frac{1}{f_j}\Big) + (n-1)\sum_{j=1}^{n} \overline{N}(r, f_j) < (\lambda + o(1))T(r, f_k), \quad \text{where } r \in I,$$

I is a set whose linear measure is infinite, $k \in \{1, 2, \ldots, n-1\}$ and $\lambda < 1$, then $f_n \equiv 1$.

Gross [4] proved that the Fermat functional equation $f(z)^2 + g(z)^2 = 1$ has the entire solutions $f(z) = \sin(h(z))$ and $g(z) = \cos(h(z))$, where $h(z)$ is any entire function, and no other solutions exist. The following lemma concerns equations with small modifications to the Fermat-type difference equations

$$f(z+c)^2 + f(z)^2 = h(z).$$

Some results on the above equation can be found in [2, 11], where $h(z)$ is an entire function with finitely many zeros or a nonzero constant.

Lemma 2.4. *Let c be a nonzero constant. All entire solutions of*

$$f(z+c)^2 - f(z)^2 = e^{-z} \tag{2.3}$$

are periodic functions with period $4i\pi$, $2c$ or $4c$.

Remark 2.5. Consider the following equation:

$$\left(\frac{f(z+c)}{e^{-z/n}}\right)^n + \left(\sqrt[n]{-1}\,\frac{f(z)}{e^{-z/n}}\right)^n = 1. \tag{2.4}$$

If $n \geq 3$, Yang's result [17, Theorem 1] shows that (2.4) has no entire solutions. If $n = 1$ in (2.4), then $f(z) = H(z) + f_1(z)$, where $H(z)$ is a periodic function with period c and $f_1(z)$ is a special solution of $f(z+c) - f(z) = e^{-z}$. This equation has entire solutions, but not all of them are periodic functions with period c. For example, $f_1(z) = (\alpha e/(1-\alpha e))e^{-z}$ with $c = 2k\pi i + \ln \alpha + 1$, $\alpha \neq 0, 1/e$, $k \in \mathbb{Z}$ is not c-periodic. Further details on finite-order transcendental entire solutions of (2.3) can be found in [2].

Remark 2.6. We recall the definition of a quasi-periodic entire function F with module g, that is, F satisfies $F(z+\tau) - F(z) = g(z)$. Chuang and Yang [3, Theorem 3.3] showed that, if $F(z+\tau) - F(z) = h(z)$, where $F(z) = f \circ g$ and $h(z)$ is a polynomial or $\rho(h) \leq 1$, then $g(z) = H_1(z) + q(z)e^{H_2(z)+Cz}$, where $H_1(z), H_2(z)$ are periodic functions with period τ, C is a constant and $q(z)$ is a polynomial. The above result is also related to the entire solution of (2.3) by taking $F(z) = f(z)^2$, which takes (2.3) into the form $F(z+\tau) - F(z) = e^{-z}$. However, this result does not give information on the periodicity of $f(z)$.

Proof of Lemma 2.4. Using Gross' result stated above,

$$\frac{f(z+c)}{e^{-z/2}} = \sin(h(z)), \qquad \frac{if(z)}{e^{-z/2}} = \cos(h(z)), \tag{2.5}$$

where $h(z)$ is any entire function such that $\sin h(z) \cos h(z) \not\equiv 0$. A basic computation from (2.5) shows that

$$e^{-c/2} \sin\Big(h(z+c) + \frac{\pi}{2} + 2k\pi\Big) = i \sin h(z), \quad k \in \mathbb{Z},$$

and hence

$$\frac{e^{-c/2}}{i} e^{i(h(z+c)+\frac{1}{2}\pi+2k\pi-h(z))} - \frac{e^{-c/2}}{i} e^{-i(h(z+c)+\frac{1}{2}\pi+2k\pi+h(z))} + e^{-2ih(z)} = 1. \tag{2.6}$$

Case 1. If $h(z)$ is a constant h, then h satisfies $e^{-2ih} = (e^{c/2}-1)/(e^{c/2}+1)$ $(\neq 0, 1, -1)$ by (2.6). Thus, $f(z) = -ie^{-z/2}\cos h$, a periodic function with period $4i\pi$.

Case 2. If $h(z)$ is not a constant, then $e^{-2ih(z)}$ is not a constant, and both $h(z+c) + h(z)$ and $h(z+c) - h(z)$ are not constants at the same time. Using Lemma 2.3, we discuss the following two subcases.

Subcase 2.1. If

$$\frac{e^{-c/2}}{i} e^{i(h(z+c)+\frac{1}{2}\pi+2k\pi-h(z))} \equiv 1, \quad -\frac{e^{-c/2}}{i} e^{-i(h(z+c)+\frac{1}{2}\pi+2k\pi+h(z))} + e^{-2ih(z)} \equiv 0,$$

then $e^{-c} = -1$. By shifting the equation (2.3) forward, $f(z+2c)^2 - f(z+c)^2 = -e^{-z}$ and so $f(z+2c)^2 = f(z)^2$, which implies that $f(z)$ is a periodic function with period $2c$ or $4c$.

Subcase 2.2. If

$$-\frac{e^{-c/2}}{i}e^{-i(h(z+c)+\frac{1}{2}\pi+2k\pi+h(z))} \equiv 1, \quad \frac{e^{-c/2}}{i}e^{i(h(z+c)+\frac{1}{2}\pi+2k\pi-h(z))} + e^{-2ih(z)} \equiv 0,$$

then $e^{-c} = -1$. By the same discussion as in Subcase 2.1, it follows that $f(z)$ is a periodic function with period $2c$ or $4c$. □

3. Proofs of the theorems

PROOF OF THEOREM 1.1. Assume that $f(z)f^{(k)}(z)$ is a periodic function with period c ($\not\equiv 0$). Then

$$f(z)f^{(k)}(z) = f(z+c)f^{(k)}(z+c).$$

Substituting $f(z) = p(z)e^{h(z)} + q(z)$ into the equation above,

$$\begin{aligned} p(z)H_k(z)e^{2h(z)} &+ [q(z)H_k(z) + p(z)q^{(k)}(z)]e^{h(z)} - p(z+c)H_k(z+c)e^{2h(z+c)} \\ &- [q(z+c)H_k(z+c) + p(z+c)q^{(k)}(z+c)]e^{h(z+c)} = q(z+c)q^{(k)}(z+c) - q(z)q^{(k)}(z), \end{aligned} \tag{3.1}$$

where $H_k(z) = p(z)[h'(z)]^k + H(z)$ is a differential polynomial in $p(z)$ and $h(z)$ with the degree in $h(z)$ and its derivatives less than k. From (3.1),

$$T(r, e^{h(z)}) = T(r, e^{h(z+c)}) + S(r, e^{h(z)})$$

and so

$$T(r, h(z)) = T(r, h(z+c)) = S(r, e^{h(z)}).$$

We discuss two cases as follows.

Case 1. If $q(z)$ is a polynomial with $\deg(q(z)) \geq k$, then $q^{(k)}(z) \not\equiv 0$ and

$$q(z+c)q^{(k)}(z+c) - q(z)q^{(k)}(z) \not\equiv 0.$$

Therefore, $h(z)$ must be a constant by Lemma 2.3 and (3.1), which is a contradiction to the hypothesis that $h(z)$ is a nonconstant entire function.

Case 2. If $q(z)$ is a polynomial with $\deg(q(z)) < k$, then $q^{(k)}(z) \equiv 0$. From (3.1),

$$\frac{p(z)H_k(z)}{q(z+c)H_k(z+c)}e^{2h(z)-h(z+c)} + \frac{q(z)H_k(z)}{q(z+c)H_k(z+c)}e^{h(z)-h(z+c)} - \frac{p(z+c)}{q(z+c)}e^{h(z+c)} = 1.$$

Now $(q(z+c)/p(z+c))e^{-h(z+c)}$ is not a constant because $h(z)$ is not a constant. From Lemma 2.3, we have two subcases.

Subcase 2.1. Assume that

$$\begin{cases} \dfrac{p(z)H_k(z)}{q(z+c)H_k(z+c)}e^{2h(z)-h(z+c)} \equiv 1, \\ \dfrac{q(z)H_k(z)}{q(z+c)H_k(z+c)}e^{h(z)-h(z+c)} - \dfrac{p(z+c)}{q(z+c)}e^{h(z+c)} \equiv 0. \end{cases} \tag{3.2}$$

Then $e^{h(z)+h(z+c)} \equiv q(z)q(z+c)/p(z)p(z+c)$. This implies that $h(z+c) \equiv B - h(z)$, where B is a constant. Thus, $(p(z)H_k(z)/q(z+c)H_k(z+c))e^{3h(z)-B} \equiv 1$ from the first equation of (3.2). So, $T(r, e^{h(z)}) = S(r, e^{h(z)})$, which is impossible.

Subcase 2.2. Suppose that

$$\begin{cases} \dfrac{q(z)H_k(z)}{q(z+c)H_k(z+c)}e^{h(z)-h(z+c)} \equiv 1, \\ \dfrac{p(z)H_k(z)}{q(z+c)H_k(z+c)}e^{2h(z)-h(z+c)} - \dfrac{p(z+c)}{q(z+c)}e^{h(z+c)} \equiv 0. \end{cases} \tag{3.3}$$

Now $e^{h(z+c)-h(z)} \equiv p(z)q(z+c)/q(z)p(z+c)$. Since $p(z)$ and $q(z)$ are polynomials, this implies that $h(z+c) - h(z) \equiv 2mi\pi$, where m is an integer, and it follows that

$$\frac{q(z+c)}{q(z)} \equiv \frac{p(z+c)}{p(z)}. \tag{3.4}$$

We will show that $p(z)$ and $q(z)$ are constants. If $h(z)$ is a nonconstant polynomial, then it must be a linear polynomial and so $H_k(z)$ is also a polynomial. From the first equation of (3.3), $q(z)$ and $H_k(z)$ are constants and so $p(z)$ is also a constant. If $h(z)$ is a transcendental entire function, then $q(z)H_k(z)/q(z+c)H_k(z+c) \equiv 1$ and $h'(z) \equiv h'(z+c)$. From the second equation of (3.3),

$$[p(z)^2 - p(z+c)^2][h'(z)]^k \equiv p(z+c)H(z+c) - p(z)H(z),$$

where $H(z)$ is a differential polynomial in $h'(z)$ with polynomial coefficients and degree less than k. If $p(z)^2 - p(z+c)^2 \not\equiv 0$, using the Clunie lemma [10, Lemma 2.4.2], we get $m(r, h') = S(r, h)$, which contradicts $h(z)$ being transcendental entire. Hence, $p(z)^2 \equiv p(z+c)^2$ from (3.4) and $p(z)$ and $q(z)$ are constants. □

Proof of Theorem 1.3. Since the period of $f(z)^n f(z+\eta)$ is c, where c is a nonzero complex number, then

$$f(z+c)^n f(z+\eta+c) = f(z)^n f(z+\eta),$$

which gives

$$\frac{f(z)^n}{f(z+c)^n} = \frac{f(z+\eta+c)}{f(z+\eta)}. \tag{3.5}$$

Let $G(z) = f(z)/f(z+c)$. From (2.1) and (3.5),

$$nT(r, G) = T\Big(r, \frac{1}{G(z+\eta)}\Big) = T(r, G(z+\eta)) + O(1) = T(r, G(z)) + S(r, G),$$

which contradicts $n \geq 2$. So, $G(z)$ must be a constant A and $A^n = A^{-1}$. Thus, $A^{n+1} = 1$, that is, $f(z)^{n+1} = f(z+c)^{n+1}$, so that $f(z)$ is a periodic function with period $(n+1)c$. □

Proof of Theorem 1.5. Since $[f(z)^n f(z+\eta)]^{(k)}$ is a periodic function with period $c \neq 0$,

$$f(z+c)^n f(z+\eta+c) = f(z)^n f(z+\eta) + P(z),$$

where $P(z)$ is a polynomial with $\deg P(z) \le k-1$. We will prove that $P(z) \equiv 0$. Since $f(z)$ is a transcendental entire function with $\rho_2(f) < 1$, Lemma 2.1 and the second main theorem for three small functions [8, Theorem 2.5] imply that

$$\begin{aligned}(n+1)T(r,f) &= T(r, f(z)^n f(z+\eta)) + S(r,f)\\ &\le \overline{N}(r, f(z)^n f(z+\eta)) + \overline{N}\Big(r, \frac{1}{f(z)^n f(z+\eta)}\Big)\\ &\quad + \overline{N}\Big(r, \frac{1}{f(z)^n f(z+\eta) + P(z)}\Big) + S(r,f)\\ &\le \overline{N}\Big(r, \frac{1}{f(z)^n f(z+\eta)}\Big) + \overline{N}\Big(r, \frac{1}{f(z+c)^n f(z+\eta+c)}\Big) + S(r,f)\\ &\le 4T(r,f) + S(r,f),\end{aligned}$$

which contradicts $n \ge 4$. Thus, $P(z) \equiv 0$. The same proof as for Theorem 1.3 can now be applied to show that f is a periodic function with period $(n+1)c$. □

Proof of Theorem 1.6. Since $[f(z)^n f(z+\eta)]^{(k)}$ is a periodic function with period η,

$$f(z)^n f(z+\eta) = f(z+\eta)^n f(z+2\eta) + P(z),$$

where $P(z)$ is a polynomial with $\deg P(z) \le k-1$. Assume that $P(z) \not\equiv 0$. Then

$$f(z+\eta)[f(z)^n - f(z+\eta)^{n-1} f(z+2\eta)] = P(z).$$

Hence,

$$f(z+\eta) = P_1(z)e^{h(z)}, \quad f(z)^n - f(z+\eta)^{n-1} f(z+2\eta) = P_2(z)e^{-h(z)},$$

where $P_1(z)P_2(z) = P(z)$ and $P_1(z)$, $P_2(z)$ are nonzero polynomials. Hence,

$$P_1(z-\eta)^n e^{nh(z-\eta)} - P_1(z)^{n-1} P_1(z+\eta) e^{(n-1)h(z)+h(z+\eta)} = P_2(z)e^{-h(z)},$$

that is,

$$P_1(z-\eta)^n e^{nh(z-\eta)+h(z)} - P_1(z)^{n-1} P_1(z+\eta) e^{nh(z)+h(z+\eta)} = P_2(z). \tag{3.6}$$

Let $f_1 := P_1(z-\eta)^n e^{nh(z-\eta)+h(z)}$ and $f_2 := -P_1(z)^{n-1} P_1(z+\eta) e^{nh(z)+h(z+\eta)}$. Then (3.6) implies that $f_1(z) + f_2(z) = P_2(z)$. If f_1 and f_2 are transcendental, using the second main theorem for three small functions [8, Theorem 2.5],

$$T(r, f_1) \le N(r, f_1) + N\Big(r, \frac{1}{f_1}\Big) + N\Big(r, \frac{1}{f_1 - P_2(z)}\Big) + S(r, f_1) \le S(r, f_1),$$

which is impossible. Thus, f_1 and f_2 are polynomials and

$$nh(z-\eta) + h(z) = nh(z) + h(z+\eta) = B, \tag{3.7}$$

where B is a constant.

If $f(z)$ is of finite order and $n \geq 1$, then $h(z)$ is a nonconstant polynomial and we have a contradiction from (3.7), so $P(z) \equiv 0$. As in the proof of Theorem 1.3, it follows that f is a periodic function with period $(n+1)\eta$.

If $f(z)$ is of infinite order and $n = 1$, then $h(z)$ may be a periodic function with period 2η. The condition $k = 1$ implies that $P(z)$ and $P_1(z)$ are constants. Thus, $f(z) = P_1 e^{h(z-\eta)}$ is a periodic function and P_1 is a nonzero constant. □

Proof of Theorem 1.8. Since the period of $[f(z)^n \Delta_\eta f]^{(k)}$ is η, where η is a nonzero complex number,

$$f(z+\eta)^n[f(z+2\eta) - f(z+\eta)] = f(z)^n[f(z+\eta) - f(z)] + Q(z),$$

where $Q(z)$ is a polynomial with $\deg Q(z) \leq k-1$. If $Q(z) \not\equiv 0$, then from the first and the second main theorems for three small functions [8, Theorem 2.5] and (2.2),

$$\begin{aligned}
nT(r,f) &\leq T(r, f(z)^n[f(z+\eta) - f(z)]) + S(r,f) \\
&\leq \overline{N}(r, f(z)^n[f(z+\eta) - f(z)]) + \overline{N}\Big(r, \frac{1}{f(z)^n[f(z+\eta) - f(z)]}\Big) \\
&\quad + \overline{N}\Big(r, \frac{1}{f(z)^n[f(z+\eta) - f(z)] + Q(z)}\Big) + S(r,f) \\
&\leq \overline{N}\Big(r, \frac{1}{f(z)}\Big) + \overline{N}\Big(r, \frac{1}{f(z+\eta) - f(z)}\Big) + \overline{N}\Big(r, \frac{1}{f(z+\eta)}\Big) \\
&\quad + \overline{N}\Big(r, \frac{1}{f(z+2\eta) - f(z+\eta)}\Big) + S(r,f) \\
&\leq \overline{N}\Big(r, \frac{1}{f(z)}\Big) + T(r, f(z+\eta) - f(z)) + \overline{N}\Big(r, \frac{1}{f(z+\eta)}\Big) \\
&\quad + T(r, f(z+2\eta) - f(z+\eta)) + S(r,f) \\
&\leq 4T(r,f) + S(r,f).
\end{aligned}$$

This contradicts $n \geq 5$, so $Q(z) \equiv 0$ and

$$f(z+\eta)^n[f(z+2\eta) - f(z+\eta)] = f(z)^n[f(z+\eta) - f(z)].$$

If $f(z+2\eta) - f(z+\eta) \equiv 0$, then $f(z)$ is a periodic function with period η. If $f(z+2\eta) - f(z+\eta) \not\equiv 0$, then

$$\frac{f(z+\eta)^n}{f(z)^n} = \frac{f(z+\eta) - f(z)}{f(z+2\eta) - f(z+\eta)} = \frac{\frac{f(z+\eta)}{f(z)} - 1}{\frac{f(z+2\eta)}{f(z)} - \frac{f(z+\eta)}{f(z)}} = \frac{\frac{f(z+\eta)}{f(z)} - 1}{\frac{f(z+2\eta)}{f(z+\eta)}\frac{f(z+\eta)}{f(z)} - \frac{f(z+\eta)}{f(z)}}.$$

Let $G(z) = f(z+\eta)/f(z)$. Then

$$nT(r, G(z)) \leq 2T(r, G(z)) + T(r, G(z+\eta)) \leq 3T(r, G(z)) + S(r, G).$$

Since $n \geq 5$, this is again a contradiction. So, $G(z)$ should be a constant $A (\neq 1)$ and $A^n = (A-1)/(A^2 - A) = 1/A$, so $A^{n+1} = 1$, that is, $f(z)^{n+1} = f(z+\eta)^{n+1}$, and $f(z)$ is a periodic function with period $(n+1)\eta$. □

Proof of Theorem 1.9. Since $(f(z)^2)^{(k)} + (f(z)^2)^{(l)}$ is a periodic function with period $c(\neq 0)$,

$$(f(z+c)^2)^{(k)} + (f(z+c)^2)^{(l)} = (f(z)^2)^{(k)} + (f(z)^2)^{(l)}. \tag{3.8}$$

We set

$$f(z+c)^2 - f(z)^2 := F(z). \tag{3.9}$$

Then (3.8) can be written as

$$F^{(k)}(z) = -F^{(l)}(z). \tag{3.10}$$

We discuss two cases as follows.

Case 1. If $k = 1$ and $l = 0$, by integrating (3.10), we have $F(z) = Ce^{-z}$, where C is a nonzero constant or $F(z) = 0$. If $F(z) = Ce^{-z}$, Lemma 2.4 implies that $f(z)$ is a periodic function with period $4i\pi$, $2c$ or $4c$. If $F(z) = 0$, then $f(z)$ is a periodic function with period c or $2c$.

Case 2. If $\rho(f) \geq 2$ and $k > l$, from (3.10), it follows that $F(z)$ must be an exponential polynomial satisfying $F^{(l)}(z) = \mu_1 e^{\lambda_1 z} + \cdots + \mu_{k-l} e^{\lambda_{k-l} z}$, where $\lambda_i^{k-l} = -1$ and the μ_i are constants for $i = 1, 2, \ldots, k-l$. Thus, $\rho(F(z)) \leq 1$.

We claim that $\rho(f(z+c) - f(z)) = \rho(f(z+c) + f(z)) = \rho(f) \geq 2$. On the one hand, $\rho(f(z+c) - f(z)) < 2$ and $\rho(f(z+c) + f(z)) < 2$ cannot both happen simultaneously, otherwise $\rho(f) < 2$, a contradiction. On the other hand, only one of $\rho(f(z+c) - f(z))$ and $\rho(f(z+c) + f(z))$ less than 2 cannot happen, otherwise $\rho(F(z)) \geq 2$, which is a contradiction. Thus, $\rho(f(z+c) - f(z)) = \rho(f(z+c) + f(z)) \geq 2$, by (3.9). From

$$2f(z) = f(z+c) + f(z) - (f(z+c) - f(z)),$$

we then have $\rho(f) \leq \rho(f(z+c) - f(z)) = \rho(f(z+c) + f(z))$. Combining the above with $\rho(f(z+c) - f(z)) \leq \rho(f)$ proves the claim.

If $F(z) \not\equiv 0$, from (3.9) and the Hadamard factorisation theorem,

$$f(z+c) - f(z) = h_1(z)e^{H(z)}, \quad f(z+c) + f(z) = h_2(z)e^{-H(z)}, \tag{3.11}$$

where $\max\{\rho(h_1), \rho(h_2)\} \leq 1$ and $\rho(e^H) \geq 2$. Thus, $T(r, h_i) = S(r, e^H)$, $i = 1, 2$. Then

$$f(z) = \tfrac{1}{2}(h_2(z)e^{-H(z)} - h_1(z)e^{H(z)}), \quad f(z+c) = \tfrac{1}{2}(h_2(z)e^{-H(z)} + h_1(z)e^{H(z)}).$$

Hence,

$$h_2(z)e^{-H(z)} + h_1(z)e^{H(z)} = h_2(z+c)e^{-H(z+c)} - h_1(z+c)e^{H(z+c)}. \tag{3.12}$$

Dividing (3.12) by $h_1(z)e^{H(z)}$,

$$f_1 + f_2 + f_3 = 1,$$

where we define $f_1 = -(h_2(z)/h_1(z))e^{-2H(z)}$, $f_2 = (h_2(z+c)/h_1(z))e^{-H(z+c)-H(z)}$ and $f_3 = -(h_1(z+c)/h_1(z))e^{H(z+c)-H(z)}$. Obviously, $-H(z+c) - H(z)$ and $H(z+c) - H(z)$ are not constants at the same time.

If $-H(z+c) - H(z)$ is not a constant, from Lemma 2.3, $f_3 \equiv 1$ and immediately

$$h_1(z)e^{H(z)} \equiv -h_1(z+c)e^{H(z+c)}. \tag{3.13}$$

From the first equation of (3.11) and (3.13),

$$f(z+c) - f(z) \equiv -(f(z+2c) - f(z+c)).$$

Thus, $f(z) \equiv f(z+2c)$ and f is a periodic function with period $2c$.

If $H(z+c) - H(z)$ is not a constant, from Lemma 2.3, $f_2 \equiv 1$ and

$$h_1(z)e^{H(z)} \equiv h_2(z+c)e^{-H(z+c)}. \tag{3.14}$$

From (3.11) and (3.14),

$$f(z+c) - f(z) \equiv f(z+2c) + f(z+c).$$

Thus, $f(z) \equiv f(z+4c)$ and f is a periodic function with period $4c$. □

4. Discussion

We have considered the periodicity of transcendental entire functions mainly under the condition $\rho_2(f) < 1$. By a careful examination of the proofs of our main results, it follows that Theorem 1.3 is also valid for transcendental meromorphic functions with $\rho_2(f) < 1$. In addition, Theorem 1.5 is true for transcendental meromorphic functions with $\rho_2(f) < 1$ and $n \geq 8$, as can be seen by appropriate application of the inequality

$$T(r, f(z)^n f(z+\eta)) \geq (n-1)T(r, f) + S(r, f), \quad \eta \in \mathbb{C} \setminus \{0\}$$

(see [12, Lemma 2.5]) in the proof of Theorem 1.5. Theorem 1.8 is valid for transcendental meromorphic functions with $\rho_2(f) < 1$ and $n \geq 10$, by using

$$T(r, f(z)^n[f(z+\eta) - f(z)]) \geq (n-1)T(r, f) + S(r, f), \quad \eta \in \mathbb{C} \setminus \{0\}$$

(see [12, Lemma 2.7]) in the proof of Theorem 1.8. The other theorems cannot be directly extended to transcendental meromorphic functions in the same way.

Acknowledgements

The authors would like to thank the referee and Professor Kai Liu for their helpful suggestions and comments.

References

[1] I. N. Baker, 'On some results of A. Rényi and C. Rényi concerning periodic entire functions', *Acta Sci. Math. (Szeged)* **27** (1966), 197–200.
[2] W. Chen, P. C. Hu and Y. Y. Zhang, 'On solutions to some nonlinear difference and differential equations', *J. Korean Math. Soc.* **53**(4) (2016), 835–846.
[3] C. T. Chuang and C. C. Yang, *Fix-Points and Factorization of Meromorphic Functions* (World Scientific, Singapore, 1990).
[4] F. Gross, 'On the equation $f^n + g^n = h^n$', *Amer. Math. Monthly* **73** (1966), 1093–1096.
[5] F. Gross and C. C. Yang, 'On periodic entire functions', *Rend. Circ. Mat. Palermo* **21**(3) (1972), 284–292.
[6] G. Halász, 'On the periodicity of composed integral functions', *Period. Math. Hungar.* **2** (1972), 73–83.

[7] R. G. Halburd, R. J. Korhonen and K. Tohge, 'Holomorphic curves with shift-invariant hyperplane preimages', *Trans. Amer. Math. Soc.* **366** (2014), 4267–4298.
[8] W. K. Hayman, *Meromorphic Functions* (Clarendon Press, Oxford, 1964).
[9] J. Heittokangas, R. Korhonen, I. Laine, J. Rieppo and J. L. Zhang, 'Value sharing results for shifts of meromorphic functions, and sufficient conditions for periodicity', *Math. Anal. Appl.* **355** (2009), 352–363.
[10] I. Laine, *Nevanlinna Theory and Complex Differential Equations* (Walter de Gruyter, Berlin–New York, 1993).
[11] K. Liu, T. B. Cao and H. Z. Cao, 'Entire solutions of Fermat type differential-difference equations', *Arch. Math.* **99** (2012), 147–155.
[12] K. Liu, X. L. Liu and L. Z. Yang, 'The zero distribution and uniqueness of difference-differential polynomials', *Ann. Polon. Math.* **109** (2013), 137–152.
[13] K. Liu and P. Y. Yu, 'A note on the periodicity of entire functions', *Bull. Aust. Math. Soc.* **100**(2) (2019), 290–296.
[14] M. Ozawa, 'On the existence of prime periodic entire functions', *Kodai Math. Sem. Rep.* **29** (1978), 308–321.
[15] A. Rényi and C. Rényi, 'Some remarks on periodic entire functions', *J. Anal. Math.* **14**(1) (1965), 303–310.
[16] Q. Wang and P. C. Hu, 'On zeros and periodicity of entire functions', *Acta Math. Sci.* **38A**(2) (2018), 209–214.
[17] C. C. Yang, 'A generalization of a theorem of P. Montel on entire functions', *Proc. Amer. Math. Soc.* **26** (1970), 332–334.
[18] C. C. Yang, 'On periodicity of entire functions', *Proc. Amer. Math. Soc.* **43** (1974), 353–356.
[19] C. C. Yang and H. X. Yi, *Uniqueness Theory of Meromorphic Functions* (Springer, Dordrecht, 2003).

XINLING LIU, Department of Mathematics,
Nanchang University, Nanchang,
Jiangxi, 330031, PR China
and
Department of Physics and Mathematics,
University of Eastern Finland, PO Box 111,
80101, Joensuu, Finland
e-mail: liuxinling@ncu.edu.cn

RISTO KORHONEN, Department of Physics and Mathematics,
University of Eastern Finland, PO Box 111,
80101, Joensuu, Finland
e-mail: risto.korhonen@uef.fi